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THE

DOCTRINE OF PROPORTION

CLEARLY DEVELOPED,

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VERY EASY SYSTEM;

OR,

THE FIFTH BOOK OF EUCLID

SIMPLIFIED.

BY

OLIVER BYRNE,
PROFESSOR OF MATHEMATICS, COLLEGE FOR CIVIL ENGINEERS; CONSULTING
ACTUARY TO THE PHILANTHROPIC LIFE ASSURANCE SOCIETY.
AUTHOR OF
"THE NEW AND IMPROVED SYSTEM OF LOGARITHMS;" "THE PRACTICAL, COMPLETE,
AND CORRECT GAGER;" "A TREATISE ON SPHERICAL TRIGONOMETRY;" "THE
ELEMENTS OF EUCLID, BY COLOURS;" "HOW TO MEASURE THE
LENGTH OF A DEGREE ON THE EARTH'S SURFACE, WITH
THE ASSISTANCE OF RAILROADS;" ETC., ETC.
INVENTOR OF
"THE GAGER'S PATENT CALCULATING INSTRUMENTS;" "THE SYSTEM OF FACILITATING THE
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TO

THE PRESIDENT, VICE-PRESIDENTS,

COUNCIL, AND STUDENTS,

OF THE

COLLEGE FOR CIVIL ENGINEERS,

THIS WORK IS DEDICATED,

BY

THEIR MOST OBEDIENT SERVANT,

THE AUTHOR.

PREFACE.

PREFACES are seldom read, because they are oftener written to magnify the merits of the books they introduce, or the doctrines which they advocate; to point out the defects, or run down the merits of others; to gratify the pride of an author, in showing the public how much he knows of the subject upon which he writes; to forward the views of publishers, in creating a sale for their publications, rather than to prepare the mind for the reception of the subject about to be introduced, by giving a general view of the matter treated of, and the manner in which it is treated, or to set the reader on his guard, that he may steer clear of the difficulties and obstructions which may retard him in his progress, as all proemial matter ought to do. For these latter purposes alone this preface is written. Therefore all who intend to study these pages would do well to read attentively the following directions and observations; for the subject upon which they are written is considered one really difficult.

Why it should be considered so, will readily be conceived when such men as Legendre, Leslie, Keith, Bonnycastle, Austin, Brewster, Young, and in fact every one who has attempted to treat the doctrine of Geometrical Proportion on any plan differing from Euclid's, have committed errors, overlooked mistakes, retrenched the generality of Euclid's reasonings, fallen into logical absurdities, or confined the general application of a subject which pervades a whole course of mathematics; while there is not one mistake, oversight, or logical objection in the whole of Euclid's Fifth Book. "In fact, Euclid's Fifth Book is a master-piece of human reasoning."

Censure on the works of others should be avoided as much as possible, because it shows the want of knowledge; those who know least, censure most: to correct a copy is easier than to produce an original; for men acquire criticism before ability, and it is mostly from those who possess no judgment that the most sweeping judgment comes.

But we wish to impress on the reader not to consider for a moment that, while we thus point out the defects of others, any wish is entertained to detract from the well-earned reputation of such men as are here mentioned; for we should rather praise them for their worth, and admire and adopt their beauties, than condemn them for a few faults: any attempt on our part to detract from the merits of such men would be presumptuous, arrogant, and unjust. We point out their *oversights* and *mistakes* alone, to imprint on the mind of the young mathematical student the necessity of close reasoning, and for the purpose of showing the consequences of coming to hurried and undigested conclusions; also, that the student might be made aware that a difficulty does exist: the nature of such difficulty he should likewise know; that he might, by the consideration of a sufficient number of examples, acquire confidence in the results of his demonstrations.

Legendre, *the great French geometer*, does not give proportion a place in his Elements of Geometry, for he was of opinion that the subject belonged to arithmetic, and algebra, not to geometry at all. Now this opinion must be totally wrong, for it is almost universally allowed, that the most refined specimen of human reasoning is to be found in Euclid's doctrine of geometrical proportion; while no such concession can be made in favour of the subject when treated of either algebraically or arithmetically. It has been justly observed, that "any system of geometry made less by geometrical proportion must be miserably defective."

Sir David Brewster, in his Translation of Legendre's Geometry, falls into a notable error, insomuch that he makes assertions which are not at all true. In speaking of Legendre, he says, "the author has provided for the application of proportion to incommensurable quantities, and demonstrated every case of this kind as it occurred, by means of the *reductio ad absurdum*." This assertion Professor Young very justly questioned, and has given examples from Brewster's translation, where the inference does not hold good.

Professor Leslie's "Elements of Geometry" is remarkable for false demonstrations; and in his fifth book he demonstrates the propositions on proportion to be true, when the magnitudes are commensurable. The fact, that those demonstrations do not hold good when the magnitudes are incommensurable, seems well known to Mr. Leslie; but that those magnitudes should also be homogeneous is altogether neglected by the learned professor.

Mr. Keith, desirous of applying a new demonstration to Proposition XVIII, in his edition of Euclid, falls into an egregious error, as he employs alternation to quantities whose antecedents might be heterogeneous: this mistake appears to be very common among Euclid's modern *improvers*, and to account why it is so is rather difficult; but a deviation from truth is easily committed, as there is but *one* truth, and many *seeming* truths.

Bonnycastle, in two of the principal propositions of his fifth book, gives demonstrations undoubtedly intended for general ones, which only apply to cases where all the magnitudes are of the same kind. That those demonstrations were intended for general ones there can be no doubt; for in his notes, page 257, Bonnycastle finds fault with Euclid's method of composition and division of ratios, as not being sufficiently general; and quotes Thomas Simpson as *very properly* making this remark: however, this assertion of Bonnycastle, both with respect to T. Simpson and Euclid, has been very justly contradicted by Professor Young, after the lapse of thirty years.

Dr. Austin, in his "Examination of Euclid," commits the same error as Mr. Keith, that of allowing demonstrations which only apply to particular cases to be substituted for general ones: most probably Keith adopted his demonstration from Austin, who recommends it in a very high degree; yet it is surprising how such errors can exist in the writings of such men, or how one can copy them from another without detecting them; since to bear in mind that quantities of a different kind can have no ratio to each other would have prevented such oversights.

That able mathematician and ingenious elementary writer, Professor Young, who so ably criticised our modern writers on geometry, especially on this subject, in cultivating the ideas of M. da Cunha, falls himself, if not into an error, into a very great inconsistency—that of discarding Euclid's doctrine of ratios from his fifth book. He undoubtedly treats of geometrical proportion without using the term ratio; but he gives other terms of a more lengthened nature, which precisely convey the same meaning: now, to do away with ratio here, is to do away with it in every subject that follows, or through a whole course of mathematics; and any such attempt should not be entertained, for it is not so very difficult to define what is intended to be expressed by the term ratio.

It is a different thing to have a clear conception of what the technical term "ratio" is meant to convey, from knowing that what is intended by the term cannot be exactly expressed in many cases by numbers. Professor Young will not deny (for they are his own words) that "the term in reality denotes the quotient arising from the division of one magnitude or quantity by another of the same kind (or the multiple or submultiple which an antecedent is of its consequent); it is accurately assignable (in numbers) when the magnitudes are commensurable, but unassignable (in numbers) when they are incommensurable." When this simple fact is known, what is to be understood by the term cannot be misconstrued, although we do allow that in many cases the exact ratio of one magnitude to another of the same kind cannot be expressed by numbers; this may be a fault in our present system of notation, or in the plan adopted for finding a common measure, and not in our geometrical notion of that which is to be conveyed by the term. And the impossibility of a person rightly understanding what is meant by saying, as A is to B, so is C to D, without embodying the idea of what is here expressed by the term *ratio* cannot be denied; it matters not by what phrase, word, form, or mode of language the idea is conveyed to the mind. The student will readily perceive that the term *ratio* is not intended to convey a real and substantial essence, but merely a simple conception of the mind, which can be well defined, and not, as some writers would have it, an "ill-defined, or unknown term." No; it is a term so interwoven through almost every part of mathematics, that to expunge it would be almost impossible; nor can any real good proceed from substituting another term in its stead.

These strictures could have been carried much further: indeed, they might include many mathematical writers of much more repute than those alluded to, and several of a minor consideration, whose names should not be ranked among geometers; in the former class Dr. Simpson, the great restorer of ancient geometry, would not be exempt; nor even *Newton*, for in the 17th lemma of his "Principia," edit. 1713, and in other places, he uses *given ratios*, and *ratios that are always the same*, for one and the same thing; but such mistakes should not be admitted, as they may lead to other errors. Among the latter class, above referred to, may be mentioned the author of a tract entitled, "The Connection of Number and Magnitude," which is a curious mixture of "good and evil." To a young student it would be difficult to determine what doctrine the Author advocates, as he seems to be one of those who, to guard against objections, take shelter in obscurity, and leave the meaning doubtful.

Why so many writers on geometry wish to depart from Euclid's method of treating proportion is hard to be accounted for, yet how few of them have a correct notion of geometrical proportion! it may partly arise from their unwillingness to acknowledge that Euclid, nearly 2,130 years ago, had arrived at an ultimate stage of perfection; and they would feign set up some contrivance to show that their powers of improvement were not exhausted.

While we uphold the opinion that it is indispensably necessary to treat the doctrine of proportion geometrically, and so highly esteem Euclid's method, let it not be supposed that we condemn the arithmetical or algebraical system of treatment, for we are convinced that the doctrine of proportion should be fully and fairly developed in every complete elements of arithmetic, of algebra, and of geometry; because it is a subject which belongs to one as much as to either of the other two; and on these three elementary sciences a whole course of mathematics is founded, through the entire of which the doctrine of proportion is mingled, and it matters not whether the structure be raised upon numbers, symbols, or lines, &c., the well-being of the combination must wholly depend on the capacity and firmness of the foundations.

For if we question the propriety of letting x, y, z, &c., stand for magnitudes, regardless whether they be incommensurable or not, we must question the whole of our beautiful system of analysis, which is just as certain in its results as plane geometry, and much more extensive in its application.

Nor do we defend the system of showing how inadequate numbers or letters are to express magnitudes, or their relations to one another; for it is admirable to see how the parallel propositions agree, although managed by means essentially different. To thoroughly understand the doctrine of proportion, it should first be acquired arithmetically, then algebraically, and both these methods made subservient to the right understanding the developement of geometrical proportion.

The introduction of symbols into works on geometry is every day becoming more general; and as by their assistance the demonstrations can be more perspicuously arranged, and the train of arguments exhibited more systematic and concise, it would therefore be unnecessary to offer any remark on their adoption in the present performance; besides, symbols, while recording each stage of the proposition faithfully, relieve the mind to contemplate the absolute quantities. But the symbols used in geometry must be considered not only as appropriate emblems of the quantities themselves, but also as expressive; and not as any measures or numerical values of them.

However, lest the student should confound these visible symbols with the abstractions for which they stand, let us take A × B in a geometrical sense: now we have no idea of the product of two numbers, but of a real rectangular space, comprehended under two right lines, represented by A and B, with two others equal to them, to complete the parallelogram.

Nor is $\frac{B \times C}{A}$ to be understood in the light of an algebraic fraction, but as a right line which is a fourth proportional to three other right lines, which are represented by A, B, and C. And when we say, ○ : □ :: ◇ : △, we do not annex the idea to those, that the circle is to the square as the rhombus is to the sector; no, these quantities may represent any magnitude whatever, whose antecedents and consequents are homogeneous abstractedly considered: but in pure geometry regard is always had to lines, surfaces, or solids.

With respect to the explanation of signs used in this work, they need but little, as they are all in common use in algebraic notation; however, to be more particular, let us take a line from the demonstration of Prop. VII, viz.:—

“∴ if M ○ ⊏, =, or ⊐ m □, then M ◇ ⊏, =, or ⊐ m □.”

This in ordinary language would be thus expressed: therefore if M times the magnitude represented by ○ be greater, equal, or less than *m* times the magnitude represented by □, then will M times the magnitude represented by ◇ be greater, equal, or less than *m* times the magnitude represented by □. When these distinctions are understood, the method of expressing demonstrations symbolically will be equally logical, strict, and convincing, without being attended with that tediousness and circuitous detail which frequently accompany other methods.

Should the symbols which represent each magnitude be differently coloured, it would greatly aid the student in the demonstrations. This system was originally intended, but abandoned, on account of the great expense of printing in colours: the want may be easily supplied by the learner.

One particular more may be worthy of remark, that is, with respect to the fifth definition, which has ever been a stumbling-block to students commencing the fifth book, and a source of much controversy and dispute among mathematicians, both in ancient and modern times. In this however we have adopted some slight modification in the words of the original text, but not the slightest change in the nature of the definition; the alteration principally consists in the adoption of "*every* equimultiple," instead of "*any* equimultiple whatever;" for most beginners form a notion, from the last sentence, that *any* promiscuously-chosen equimultiple whatever would be sufficient to test whether four magnitudes were proportional or not; and seldom conceives that the conditions of the definition require to be fulfilled with every set of equimultiples that might be selected. This test for proportionals has been regarded by some as a paradox, or as a thing impossible to be applied, and by others quite foreign to the purpose; but it does not follow from this definition that an infinite number of trials must be made every time we want to test four proportionals; we only want to establish this as a standard, and when once allowed the difficulty is at once removed. This we presume will be fully established and readily comprehended from our enunciation of this definition.

Innumerable have been the attempts to elude or surmount the obstacle contained in this definition, but not one of those have been more successful than another, and on a mature consideration it is evident that no other definition essentially different could have been given equally applicable and general; and it is very probable that even those definitions, which since the time of Euclid have been proposed as substitutes for the fifth, presented themselves to him, and that from their want of generality he was obliged to reject them. It is true, definitions should require no explanation, nor should they contain words which need themselves to be defined —they should be clear and perspicuous: notwithstanding this we have explained many of the definitions more familiarly, lest the most ordinary capacity should fail to comprehend them. To teach should be the highest aim of a writer on elementary subjects, and not to adopt (which is too often the case) that stiff and formal manner so prejudicial to and inconsistent with the ideas of a learner; every thing likely to embarrass should be explained, and that authorial kind of scientific dignity should be set aside when the object is to instruct others.

The following are the objects for which this work is published:—to uphold Euclid's fifth book as the only legitimate doctrine of geometrical proportion as yet produced; to show that proportion should be treated of algebraically and arithmetically as well as geometrically, as it equally belongs to all; and to endeavour to clear, without destroying the universality and rigour of its conclusions, this extensive mathematical branch of that difficult, elaborate, and intricate reasoning with which the prevailing opinion has so long charged it.

How far these objects are attained remains for the impartial scientific public to decide, on whose judgment alone depends its approval or condemnation.

College for Civil Engineers,
Putney, 19 *November*, 1840.

BOOK V.

DEFINITIONS.

I.

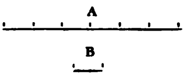
A LESS magnitude is said to be an aliquot part or submultiple of a greater magnitude, when the less measures the greater; that is, when the less is contained a certain number of times exactly in the greater.

Thus, if □□ □□ be contained in □□□□□□ □□□□□□ three times exactly, the former is said to be a submultiple of the latter. The number 5 is said to be an aliquot part or submultiple of 20, because 5 measures 20: *i.e.*, 5 is contained in 20 four times exactly; the same may be said with respect to other numbers and magnitudes.

II.

A greater magnitude is said to be a multiple of a less, when the greater is measured by the less; that is, when the greater contains the less a certain number of times exactly.

Let the line A be 6 feet, and B one foot in length; A is said to be a multiple of B, because A contains B a certain number of times exactly. Again; 24 is said to be a multiple of 2, because 24 contains 2 twelve times, without leaving a remainder.



III.

Ratio is the relation which one quantity bears to another of the same kind, with respect to magnitude.

Several definitions have been given of the term "ratio;" the following are quoted from some of the best writers on proportion:—

"Ratio is a mutual relation of two magnitudes of the same kind to one another, in respect of quantity."—*Simson's Euclid.*

"Ratio is the relation which one quantity bears to another in respect of magnitude, the comparison being made by considering what multiple part or parts one is of the other."—*Wood's Algebra.*

"Ratio is that relation of two quantities of the same kind, which arises from considering what part or parts the one is of the other."—*Bonnycastle's Geometry.*

"Ratio is usually defined to be the relation which one quantity bears to another of the same kind, with respect to magnitude: and as such relation may be expressed either by stating how much one exceeds the other, or how often one contains the other; ratio has accordingly been divided into two kinds—arithmetical ratio, and geometrical ratio.

"Arithmetical ratio is that which expresses the difference of the quantities compared.

"Geometrical ratio expresses the quotient arising from the division of the quantities compared."—*Young's Algebra.*

If there be two lines, one of which is 16 feet long, the other 3, the length of the former is said to have to that of the latter the ratio of 16 to 3; generally written, 16 : 3, in which the first term (16) is called the antecedent, and the other (3) the consequent. Or the ratio is sometimes written in the form of a fraction, thus, $\frac{16}{3}$ equal $5\frac{1}{3}$, showing that the antecedent contains the consequent $5\frac{1}{3}$ times.

The area of a parallelogram, whose length is 4 feet and breadth 3, is to the area of another, whose length is 8 feet and breadth 6, in the ratio of 12 : 48; and may be expressed by the fraction $\frac{12}{48} = \frac{1}{4}$, showing that the antecedent is $\frac{1}{4}$ of the consequent.

Let there be two cubes, each of the linear edges of the first is 4 feet, and of the second 2 feet; then the length of all the linear edges of the first is to the length of all the linear edges of the second, as 48 : 24 : the area of the whole surface of the first is to that of the second, as 96 : 24, and the solidity of the first cube is to that of the second in the ratio of 64 to 8. In common language these ratios would be expressed by saying, the first cube is eight times as big as the second, the surface of the first is three times as great as the surface of the second, and the linear sides of the first are twice as long as those of the second. But, if the first of these be a cube of water, and the other copper, the ratio of the weight of the first is to that of the second very nearly as 8 : 9, or the first is only $\frac{8}{9}$ths the weight of the second, although it is 8 times as big; this can be easily shown, as a cubic foot of water weighs 1,000 ounces, and that of copper 9,000 ounces nearly.

The term ratio is used in comparing different degrees of heat and light, and other things quite foreign to geometrical magnitudes. The doctrine of ratios, generally treated, requires not the aid of numbers, but the moment we descend to particular cases the idea of number presents itself; and, in many cases, numbers are inadequate to express exactly the ratios of geometrical magnitudes, or even the relations which exist among one another; and yet although the ratios referred to cannot be expressed exactly by numbers, they can be expressed to any designed degree of exactness; in such cases the term "ratio nearly" is applied. Of this we will give one or two instances here:—When the diameter of a circle is 1, the circumference is 3.14159265 very nearly; for although we cannot find what the true circumference is, yet we know that 3.14159265 does differ from it $\frac{1}{1000000000}$ part of a unit: from this we infer that the ratio of the diameter to the circumference is as 1 : 3.14159265 nearly. The ratio of the square root of 2, to the square root of 7, is 1.4142136 : 2.6457513 nearly. 1.4142136 is not the exact square root of 2, nor can the exact square root be obtained; but yet we may approach to it to any designed degree of exactness. The number above given does not differ $\frac{1}{10000000}$ part of a unit from the square root of 2, the same may be said of the number 7; therefore we infer that 1.4142136 : 2.6457513 expresses the "ratio nearly" of the square root of 2 to the square root of 7. The term ratio has been applied by mathematical writers to signify different relations, besides that relation which Euclid intended it to express; this has led to a great deal of confusion, and should be discontinued, or the difference shown when such term is used. Some writers differ so far from Euclid's plan, as to say, "it matters not whether we consider how often the first term contains the second, or how often the second contains the first;" now, according to the principles laid down in Euclid's Fifth Book, 12 : 3 is said to be a greater ratio than 12 : 4, because 12 contains 3 a greater number of times than 12 contains 4. Quite the contrary conclusion must be come to, if we consider how often the second term contains the first. This latter plan of comparing ratios must be instituted for the purpose of differing from Euclid, as it is not in any way superior; and besides, the disorder that must follow in the comparison of ratios, by plans so widely differing; for that which is called greater ratio by one, is a less ratio by the other.

In the arithmetical series, a, $a+r$, $a+2r$, $a+3r$, &c., r is sometimes called the common arithmetical ratio; and, in the geometrical series, a, ar, ar^2, ar^3, &c., r is called the common ratio; in the first of these series, r has no right whatever to the term ratio; and, in the second, the ratio of the first term to the second, the second to the third, the third to the fourth, is expressed by $\frac{1}{r}$, and not by r.

IV.

Magnitudes are said to have a ratio to one another, when they are of the same kind; and the one which is not the greater can be multiplied so as to exceed the other.

[The other definitions will be given throughout the book, where their aid is first required.]

AXIOMS.

I.

EQUIMULTIPLES or equisubmultiples of the same, or of equal magnitudes, are equal.

If $A = B$, then
twice A = twice B, that is,
$2\,A = 2\,B$;
$3\,A = 3\,B$;
$4\,A = 4\,B$;
&c. &c.
and $\frac{1}{2}$ of A = $\frac{1}{2}$ of B;
$\frac{1}{3}$ of A = $\frac{1}{3}$ of B;
&c. &c.

II.

A multiple of a greater magnitude is greater than the same multiple of a less.

Let $A \sqsubset B$, then
$2\,A \sqsubset 2\,B$;
$3\,A \sqsubset 3\,B$;
$4\,A \sqsubset 4\,B$;
&c., &c.

III.

That magnitude, of which a multiple is greater than the same multiple of another, is greater than the other.

Let $2\,A \sqsubset 2\,B$, then
$A \sqsubset B$;
or, let $3\,A \sqsubset 3\,B$, then
$A \sqsubset B$;
or, let $m\,A \sqsubset m\,B$, then
$A \sqsubset B$

PROP. I. THEO.

If any number of magnitudes be equimultiples of as many others, each of each: what multiple soever any one of the first is of its part, the same multiple shall of the first magnitudes taken together be of all the others taken together.

Let ●●●●● be the same multiple of ●, that ▼▼▼▼▼ is of ▼, that □□□□□ is of □.

Then it is evident that

$$\left.\begin{matrix}\bullet\bullet\bullet\bullet\bullet\\ \blacktriangledown\blacktriangledown\blacktriangledown\blacktriangledown\blacktriangledown\\ \square\square\square\square\square\end{matrix}\right\}\text{ is the same multiple of }\left\{\begin{matrix}\bullet\\ \blacktriangledown\\ \square\end{matrix}\right.$$

which that ●●●●● is of ●; because there are as many magnitudes in $\left\{\begin{matrix}\bullet\bullet\bullet\bullet\bullet\\ \blacktriangledown\blacktriangledown\blacktriangledown\blacktriangledown\blacktriangledown\\ \square\square\square\square\square\end{matrix}\right\}=\left\{\begin{matrix}\bullet\\ \blacktriangledown\\ \square\end{matrix}\right\}$ as there are in ●●●●● = ●.

The same demonstration holds in any number of magnitudes, which has here been applied to three.

∴ If any number of magnitudes, &c.

Limited Arithmetical Exposition.

7 is the same multiple of 1
that 42 . . . is of . . . 6
„ 35 . . . is of . . . 5
and that 14 . . . is of . . . 2

Then the sum of 7, 42, 35, and 14 or 98, is the same multiple of 1, 6, 7, and 2 together, that 7 is of 1, that 42 is of 6, &c., *i. e.* 98 is seven times 14.

Partial Algebraical Exposition.

$m a$ is the same multiple of a
that $m b$ . . . is of . . . b
„ $m c$ . . . is of . . . c
and that $m d$ . . . is of . . . d
&c., &c.

∴ $m a + m b + m c + m d +$,&c., or $m\ (a + b + c + d +$,&c.,) is the same multiple of $a + b + c + d +$, &c., that $m a$ is of a, $m b$ is of b, &c.

PROP. II. THEO.

If the first magnitude be the same multiple of the second that the third is of the fourth, and the fifth the same multiple of the second that the sixth is of the fourth, then shall the first, together with the fifth, be the same multiple of the second that the third, together with the sixth, is of the fourth.

Let ●●●, the first, be the same multiple of ●, the second, that △△△, the third, is of △, the fourth; and let ●●●●, the fifth, be the same multiple of ●, the second, that △△△△, the sixth, is of △, the fourth.

Then it is evident, that $\left\{\begin{matrix}\bullet\bullet\bullet\\ \bullet\bullet\bullet\bullet\end{matrix}\right\}$, the first and fifth together, is the same multiple of ●, the second, that $\left\{\begin{matrix}\triangle\triangle\triangle\\ \triangle\triangle\triangle\triangle\end{matrix}\right\}$, the third and sixth together, is of the same multiple of △, the fourth; because there are as many magnitudes in $\left\{\begin{matrix}\bullet\bullet\bullet\\ \bullet\bullet\bullet\bullet\end{matrix}\right\} = \bullet$ as there are in $\left\{\begin{matrix}\triangle\triangle\triangle\\ \triangle\triangle\triangle\triangle\end{matrix}\right\} = \triangle$.

∴ If the first magnitude, &c.

Limited Arithmetical Exposition.

20 the first, 2 second, 350 the third, 35 the fourth,
18 the fifth, 315 the sixth.

Then it is evident, that 20 and 18 together is the same multiple of 2, that 350 and 315 together is of 35.

Algebraical Exposition.

$m\,a$ the first is the same multiple of a the second that $m\,b$ the third is of b the fourth, and $n\,a$ the fifth is the same multiple of a the second, that $n\,b$ the sixth is of b the fourth; then it is evident that $m\,a + n\,a$, or $(m+n)\,a$ the first and fifth together, is the same multiple of a the second that $(m\,b + n\,b)$, or $(m+n)\,b$ the third and sixth together, is of b the fourth.

PROP. III. THEO.

If the first of four magnitudes be the same multiple of the second that the third is of the fourth, and if any equimultiples whatever of the first and third be taken, those shall be equimultiples; one of the second, and the other of the fourth.

Let $\left\{\begin{matrix}\square\\ \square\square\\ \square\end{matrix}\right\}$ be the same multiple of □ which $\left\{\begin{matrix}\blacktriangle\\ \blacktriangle\blacktriangledown\blacktriangle\end{matrix}\right\}$ is of ▲;

the same multiple of

is of

take $\left\{\begin{matrix}\square\square\square\square\\ \square\square\square\square\\ \square\square\square\square\\ \square\square\square\square\end{matrix}\right\}$ which $\left\{\begin{matrix}\blacktriangle\\ \blacktriangle\blacktriangledown\blacktriangle\\ \blacktriangledown\blacktriangle\blacktriangledown\blacktriangle\blacktriangledown\\ \blacktriangle\blacktriangledown\blacktriangle\blacktriangledown\blacktriangle\blacktriangledown\blacktriangle\end{matrix}\right\}$

Then it is evident,

that $\left\{\begin{matrix}\square\square\square\square\\ \square\square\square\square\\ \square\square\square\square\\ \square\square\square\square\end{matrix}\right\}$ is the same multiple of □ which $\left\{\begin{matrix}\blacktriangle\\ \blacktriangle\blacktriangledown\blacktriangle\\ \blacktriangledown\blacktriangle\blacktriangledown\blacktriangle\blacktriangledown\\ \blacktriangle\blacktriangledown\blacktriangle\blacktriangledown\blacktriangle\blacktriangledown\blacktriangle\end{matrix}\right.$ is of ▼; because $\left\{\begin{matrix}\square\square\square\square\\ \square\square\square\square\\ \square\square\square\square\\ \square\square\square\square\end{matrix}\right\}$ contains $\left\{\begin{matrix}\square\\ \square\square\\ \square\end{matrix}\right\}$ contains □ as many times as $\left\{\begin{matrix}\blacktriangle\\ \blacktriangle\blacktriangledown\blacktriangle\\ \blacktriangledown\blacktriangle\blacktriangledown\blacktriangle\blacktriangledown\\ \blacktriangle\blacktriangledown\blacktriangle\blacktriangledown\blacktriangle\blacktriangledown\blacktriangle\end{matrix}\right\}$ contains $\left\{\begin{matrix}\blacktriangle\\ \blacktriangle\blacktriangledown\blacktriangle\end{matrix}\right\}$ contains ▲ : and the same reasoning is applicable in all cases.

∴ If the first of four, &c.

Arithmetical Exposition.

Suppose the first of four numbers to be three times the second, and the third three times the fourth, as 33, 11, 24, and 8.

Then, if we take four times the first (132) and four times the third (96), it is evident that 132 is the same multiple of 11 the second, that 96 is of 8 the fourth, for 132 is 12 times 11, and 96 is 12 times 8.

Algebraical Exposition.

Let the four magnitudes be *m a*, *a*, *m b*, and *b*, take equimultiples of the first and third, as, *n* times the first, and *n* times the third; then it is evident that *n m a*, or *n m* times *a*, is the same multiple of *a*, that *n m b*, or *n m* times *b*, is of *b*.

DEFINITION V.

Four magnitudes, ●, □, ▲, and △, are said to be proportionals when every equimultiple of the first and third be taken, and every equimultiple of the second and fourth, as,

●●	▲▲	□□	△△
●●●	▲▲▲	□□□	△△△
●●●●	▲▲▲▲	□□□□	△△△△
●●●●●	▲▲▲▲▲	□□□□□	△△△△△
●●●●●●	▲▲▲▲▲▲	□□□□□□	△△△△△△
&c.,	&c.	&c.,	&c.

Then taking every pair of equimultiples of the first and third, and every pair of equimultiples of the second and fourth,

If
●● ⊏, = or ⊐ □□
●● ⊏, = or ⊐ □□□
●● ⊏, = or ⊐ □□□□
●● ⊏, = or ⊐ □□□□□
●● ⊏, = or ⊐ □□□□□□
then will
▲▲ ⊏, = or ⊐ △△
▲▲ ⊏, = or ⊐ △△△
▲▲ ⊏, = or ⊐ △△△△
▲▲ ⊏, = or ⊐ △△△△△
▲▲ ⊏, = or ⊐ △△△△△△

That is, if twice the first be greater, equal, or less than twice the second, twice the third will be greater, equal, or less than twice the fourth; or, if twice the first be greater, equal, or less than three times the second, twice the third will be greater, equal, or less than three times the fourth, and so on, as above expressed.

$$\text{If}\left\{\begin{array}{ll} \bullet\bullet\bullet \sqsubset, = \text{ or } \sqsupset & \square\square \\ \bullet\bullet\bullet \sqsubset, = \text{ or } \sqsupset & \square\square\square \\ \bullet\bullet\bullet \sqsubset, = \text{ or } \sqsupset & \square\square\square\square \\ \bullet\bullet\bullet \sqsubset, = \text{ or } \sqsupset & \square\square\square\square\square \\ \bullet\bullet\bullet \sqsubset, = \text{ or } \sqsupset & \square\square\square\square\square\square \end{array}\right\}\text{then will}\left\{\begin{array}{ll} \blacktriangle\blacktriangle\blacktriangle \sqsubset, = \text{ or } \sqsupset & \triangle\triangle \\ \blacktriangle\blacktriangle\blacktriangle \sqsubset, = \text{ or } \sqsupset & \triangle\triangle\triangle \\ \blacktriangle\blacktriangle\blacktriangle \sqsubset, = \text{ or } \sqsupset & \triangle\triangle\triangle\triangle \\ \blacktriangle\blacktriangle\blacktriangle \sqsubset, = \text{ or } \sqsupset & \triangle\triangle\triangle\triangle\triangle \\ \blacktriangle\blacktriangle\blacktriangle \sqsubset, = \text{ or } \sqsupset & \triangle\triangle\triangle\triangle\triangle\triangle \end{array}\right.$$

&c., &c. &c., &c.

In other terms, if three times the first be greater, equal, or less than twice the second, three times the third will be greater, equal, or less than twice the fourth; or, if three times the first be greater, equal, or less than three times the second, then will three times the third be greater, equal, or less than three times the fourth; or if three times the first be greater, equal, or less than four times the second, then will three times the third be greater, equal, or less than four times the fourth, and so on. Again,

$$\text{If}\left\{\begin{array}{ll} \bullet\bullet\bullet\bullet \sqsubset, = \text{ or } \sqsupset & \square\square \\ \bullet\bullet\bullet\bullet \sqsubset, = \text{ or } \sqsupset & \square\square\square \\ \bullet\bullet\bullet\bullet \sqsubset, = \text{ or } \sqsupset & \square\square\square\square \\ \bullet\bullet\bullet\bullet \sqsubset, = \text{ or } \sqsupset & \square\square\square\square\square \\ \bullet\bullet\bullet\bullet \sqsubset, = \text{ or } \sqsupset & \square\square\square\square\square\square \end{array}\right\}\text{then will}\left\{\begin{array}{ll} \blacktriangle\blacktriangle\blacktriangle\blacktriangle \sqsubset, = \text{ or } \sqsupset & \triangle\triangle \\ \blacktriangle\blacktriangle\blacktriangle\blacktriangle \sqsubset, = \text{ or } \sqsupset & \triangle\triangle\triangle \\ \blacktriangle\blacktriangle\blacktriangle\blacktriangle \sqsubset, = \text{ or } \sqsupset & \triangle\triangle\triangle\triangle \\ \blacktriangle\blacktriangle\blacktriangle\blacktriangle \sqsubset, = \text{ or } \sqsupset & \triangle\triangle\triangle\triangle\triangle \\ \blacktriangle\blacktriangle\blacktriangle\blacktriangle \sqsubset, = \text{ or } \sqsupset & \triangle\triangle\triangle\triangle\triangle\triangle \end{array}\right.$$

&c., &c. &c., &c.

And so on, with any other equimultiples of the four magnitudes, taken in the same manner.

Euclid expresses this definition as follows:—

The first of four magnitudes is said to have the same ratio to the second, which the third has to the fourth, when any equimultiples whatsoever of the first and third being taken, and any equimultiples whatsoever of the second and fourth; if the multiple of the first be less than that of the second, the multiple of the third is also less than that of the fourth; or, if the multiple of the first be equal to that of the second, the multiple of the third is also equal to that of the fourth; or, if the multiple of the first be greater than that of the second, the multiple of the third is also greater than that of the fourth.

In future we shall express this definition generally, thus:—

If M ● ⊏, = or ⊐ *m* □, when M ▲ ⊏, = or ⊐ *m* △.

Then we infer, that ● the first, has the same ratio to □, the second, which ▲, the third, has to △ the fourth: expressed in the succeeding demonstrations, thus:—

● : □ :: ▲ : △;

or thus, ● : □ = ▲ : △;

or thus, $\frac{\bullet}{\square} = \frac{\blacktriangle}{\triangle}$: and is read,

"as ● is to □, so is ▲ to △."

And if ● : □ :: ▲ : △ we shall infer if M ● ⊏, = or ⊐ *m* □, then will M ▲ ⊏, = or ⊐ *m* △. That is, if the first be to the second, as the third is to the fourth; then, if M times the first be greater than, equal to, or less than *m* times the second, then shall M times the third be greater than, equal to, or less than *m* times the fourth, in which M and *m* are not to be considered particular multiples, but every pair of multiples whatever; nor are such marks as ●, △, □, &c., to be considered any more than representatives of geometrical magnitudes.

The student should thoroughly understand this definition before proceeding further.

PROP. IV. THEO.

If the first of four magnitudes have the same ratio to the second, which the third has to the fourth, then any equimultiples whatever of the first and third shall have the same ratio to any equimultiples of the second and fourth; viz., the equimultiple of the first shall have the same ratio to that of the second, which the equimultiple of the third has to that of the fourth.

Let ● : □ :: ▲ : △, then 3 ● : 2 □ :: 3 ▲ : 2 △., every equimultiple of 3 ● and 3 ▲ are equimultiples of ● and ▲, and every equimultiple of 2 □ and 2 △, are equimultiples of □ and △: (3. B. V).

That is, M times 3 ● and M times 3 ▲ are equimultiples of ● and ▲, and *m* 2 □ and *m* 2 △ are equimultiples of 2 □ and 2 △; but ● : □ :: ▲ : △ (hyp.); ∴ if M 3 ● ⊏, =, or ⊐ *m* 2 □, then M 3 ▲ ⊏, =, ⊐ 2 △ (by the fifth definition), and, therefore 3 ● : 2 □ :: 3 ▲ : 2 △ (by the fifth definition).

The same reasoning holds good if any other equimultiple of the first and third be taken, and any other equimultiples of the second and fourth.

∴ If the first of four magnitudes, &c.

Arithmetical Illustration.

Let 3 : 5 :: 9 : 15 be four numbers that are proportionals; and let us multiply the first and third of these numbers by any other, say 5, and the second and fourth by any number, say 2, then we have 15 : 10 :: 45 : 30, for it is easily observed that $\frac{15}{10} = \frac{45}{30}$, or 15 contains 10 as often as 45 contains 30.

Algebraical Exposition.

Let $a : b :: c : d$, then will $m a : n b :: m c : n d$, because $a : b :: c : d$, $\frac{a}{b} = \frac{c}{d}$ multiply both sides by $\frac{m}{n}$, and we have $\frac{ma}{n b} = \frac{mc}{n d}$ $\therefore$ $ma : nb :: mc : nd$.

Cor. Likewise, if the first have the same ratio to the second, which the third has to the fourth, then also any equimultiples of the first and third have the same ratio to the second and fourth; and, in like manner, the first and the third have the same ratio to any equimultiples whatever of the second and fourth.

That is, if $a : b :: c : d$, then $m a : b :: m c : d$, and $a : n b :: c : n d$.

Because $\frac{a}{b} = \frac{c}{d} \therefore \frac{m a}{b} = \frac{m c}{d}$ by multiplying both sides by m, and $\therefore m a : b :: m c : c$; it can also be readily shown that $\frac{a}{n b} = \frac{c}{n d}$ $\therefore a : n b :: c : n d$.

PROP. V. THEO.

If one magnitude be the same multiple of another, which a magnitude taken from the first is of a magnitude taken from the other, the remainder shall be the same multiple of the remainder, that the whole is of the whole.

$$\text{Let } \begin{array}{c}\bigcirc\\ \bigcirc\bigcirc\\ \square\end{array} = M' \begin{array}{c}\blacktriangle\\ \square\end{array}$$

$$\text{and } \square = M' \square,$$

$$\therefore \begin{array}{c}\bigcirc\\ \bigcirc\bigcirc\\ \square\end{array} - \square = M' \begin{array}{c}\blacktriangle\\ \square\end{array} - M' \square,$$

$$\therefore \begin{array}{c}\bigcirc\\ \bigcirc\bigcirc\end{array} = M' \left(\begin{array}{c}\blacktriangle\\ \square\end{array} - \square\right),$$

$$\text{and } \therefore \begin{array}{c}\bigcirc\\ \bigcirc\bigcirc\end{array} = M' \blacktriangle.$$

$\therefore$ If one magnitude, &c.

As a particular arithmetical example, let us take the following:

	25	is 5 times	5
	5	is 5 times	1
Remainder	20	is 5 times	4

Algebraical Exposition.

ma is the same multiple of a that mb is of b, $\therefore$ $ma - mb$ is m times $a - b$, for $ma - mb = m(a-b)$: a supposed to be greater than b.

PROP. VI. THEO.

If two magnitudes be equimultiples of two others, and if equimultiples of these be taken from the first two, the remainder are either equal to these others, or equimultiples of them.

Let $\bigcirc\bigcirc\bigcirc\bigcirc = M' \square$ and $\square\square = M' \triangle$;

then $\bigcirc\bigcirc\bigcirc\bigcirc - m' \square = M' \square - m' \square = (M' - m') \square$,

and $\square\square - m' \triangle = M' \triangle - m' \triangle = (M' - m') \triangle$,

which are equimultiples of $\square$ and $\triangle$, and equal to $\square$ and $\triangle$, when $M' - m' = 1$.

∴ If two magnitudes be equimultiples, &c.

For this case, as a particular arithmetical example, let us take—

40 and 50 } the same multiples of { 4 and 5

From 40 take 3 times 4, the remainder is 28, and from 50 take 3 times 5, the remainder is 35; it is evident that

28 and 35 } are the same multiples of { 4 and 5

From 40 take 9 times 4, and from 50 take 9 times 5, then the remainders are 4 and 5.

Algebraically.

$m\,a$ is the same multiple of a that $m\,b$ is of b; from $m\,a$ take $n\,a$, the remainder is $m\,a - n\,a = (m - n)a$; from $m\,b$ take $n\,b$, the remainder is $m\,b - n\,b = (m - n)\,b$.

Then it is evident that $(m - n)a$ is the same multiple of a, that $(m - n)b$ is of b; and $(m - n)a = a$, and $(m - n)\,b\,c = b$, when $m - n = 1$.

PROP. A. THEO.

If the first of the four magnitudes has the same ratio to the second, which the third has to the fourth, then, if the first be greater than the second, the third is also greater than the fourth; and if equal, equal; if less, less.

Let ● : □ :: ▲ : △; therefore, by the fifth definition,
if ●● ⊏ □□, then will ▲▲ ⊏ △△;
but, if ● ⊏ □, then ●● ⊏ □□, and ▲▲ ⊏ △△,
and, ∴ ▲ ⊏ △.
Similarly, if ● =, or ⊐ □, then will ▲ =, or ⊐ △.
∴ If the first of four, &c.

DEFINITION XIV.

Geometricians make use of the technical term "Invertando," by invertion, when there are four proportionals, and it is inferred, that the second is to the first as the fourth to the third.

Let A : B :: C : D, then, by "invertando" it is inferred B : A :: D : C.

PROP. B. THEO.

If four magnitudes are proportionals, they are proportionals also when taken inversely.

Let $\boxed{1} : \boxed{2} :: \boxed{3} : \boxed{4}$,

then, inversely, $\boxed{2} : \boxed{1} :: \boxed{4} : \boxed{3}$.

If M $\boxed{1} \sqsupset m \boxed{2}$, then M $\boxed{3} \sqsupset m \boxed{4}$, by the fifth definition.

Let M $\boxed{1} \sqsupset m \boxed{2}$, that is, $m \boxed{2} \sqsubset$ M $\boxed{1}$,

∴ M $\boxed{3} \sqsupset m \boxed{4}$, or, $m \boxed{4} \sqsubset$ M $\boxed{3}$;

∴ if $m \boxed{2} \sqsubset$ M $\boxed{1}$, then will $m \boxed{4} \sqsubset$ M $\boxed{3}$.

In the same manner it may be shown,

that it $m \boxed{2} =$ or $\sqsupset$ M $\boxed{1}$,

then will $m \boxed{4} =$, or $\sqsupset$ M $\boxed{4}$;

and, therefore, by the fifth definition, we infer,

that $\boxed{2} : \boxed{1} : \boxed{4} : \boxed{3}$.

∴ If four magnitudes, &c.

PROP. C. THEO.

If the first be the same multiple of the second, or the same part of it, that the third is of the fourth; the first is to the second, as the third is to the fourth.

Let □□□□, the first, be the same multiple of ○, the second,

that ◇◇◇◇, the third, is of △, the fourth.

Then □□□□ : ○ :: ◇◇◇◇ : △.

take M □□□□, *m* ○, M ◇◇◇◇, *m* △;

because □□□□ is the same multiple of ○

that ◇◇◇◇ is of △ (according to the hypothesis);

and M □□□□ is taken the same multiple of □□□□

that M ◇◇◇◇ is of ◇◇◇◇,

∴ (according to the third proposition),

M □□□□ is the same mult. of ○ that M ◇◇◇◇ is of △.

Therefore, if M □□□□ be of ○ a greater multiple than m ○ is, then M ◇◇◇◇ is a greater multiple of ◇ than m ◇ is; that is, if M □□□□ be greater than m ○, then M ◇◇◇◇ will be greater than m ◇; in the same manner it can be shown, if M □□□□ be equal m ○, then M ◇◇◇◇ will be equal m ◇.

And, generally, if

M □□□□ ⊏, = or ⊐ m ○

then M ◇◇◇◇ will be ⊏, = or ⊐ m ◇;

∴ by the fifth definition,

□□□□ : ○ :: ◇◇◇◇ : ◇.

Next, let ○ be the same part of □□□□

that ◇ is of ◇◇◇◇.

In this case also ○ : □□□□ :: ◇ : ◇◇◇◇.

For, because

○ is the same part of □□□□ that △ is of ◇◇◇◇,

therefore □□□□ is the same multiple of ○

that ◇◇◇◇ is of △.

Therefore, by the preceding case,

□□□□ : ○ :: ◇◇◇◇ : △;

and ∴ ○ : □□□□ :: △ : ◇◇◇◇,

by proposition B.

∴ If the first be the same multiple, &c.

PROP. D. THEO.

If the first be to the second as the third to the fourth, and if the first be a multiple, or a part of the second; the third is the same multiple, or the same part of the fourth.

Let ○○○ : □ :: ◇◇◇◇ : ▽;

and first, let ○○○ be a multiple □;

◇◇◇◇ shall be the same multiple of ▽.

First.	Second.	Third.	Fourth.
○○○	□	◇◇◇◇	▽
	⌓⌓⌓	◊◊◊◊.	

Take ⌓⌓⌓ = ○○○.

Whatever multiple ○○○ is of □

take ◊◊◊◊ the same multiple of ▽,

then, because ○○○ : □ :: ◇◇◇◇ : ▽

and of the second and fourth, we have taken equimultiples,

○○○ and ◊◊◊◊, therefore (cor. 4, B. v.),

○○○ : ⌂⌂⌂ :: ◇◇◇◇ : ◊◊◊◊, but (constr.),

○○○ = ⌂⌂⌂ ∴ (A. B. v.) ◇◇◇◇ = ◊◊◊◊

and ◊◊◊◊ is the same multiple of ▽

that ○○○ is of □.

Next, let □ : ○○○ :: ▽ : ◇◇◇◇,

and also □ a part of ○○○;

then ▽ shall be the same part of ◇◇◇◇.

Inversely (B. 5.), ○○○ : □ :: ◇◇◇◇ : ▽,

but □ is a part of ○○○;

that is, ○○○ is a multiple of □;

∴ by the preceding case, ◇◇◇◇ is the same multiple of ▽

i.e., ▽ is the same part of ◇◇◇◇

that □ is of ○○○.

∴ If the first be to the second, &c.

PROP. VII. THEO.

Equal magnitudes have the same ratio to the same magnitude, and the same has the same ratio to equal magnitudes.

Let ○ = ◇ and □ any other magnitude;

then ○ : □ = ◇ : □ and □ : ○ = □ : ◇

because ○ = ◇,

∴ M○ = M◇;

∴ if M○ ⊏, = or ⊐ *m*□, then M◇ ⊏, = or ⊐ *m*□

and ∴ ○ : □ = ◇ : □ (5, def. v.).

From the foregoing reasoning it is evident

that, if *m*□ ⊏, = or ⊐ M○, then, *m*□ ⊏, = or ⊐ M◇

∴ □ : ○ = □ : ◇ (5, def. v).

∴ Equal magnitudes, &c.

DEFINITION VII

When of the equimultiples of four magnitudes (taken as in the fifth definition), the multiple of the first is greater than that of the second, but the multiple of the third is not greater than the multiple of the fourth; then the first is said to have to the second a greater ratio than the third magnitude has to the fourth: and, on the contrary, the third is said to have to the fourth a less ratio than the first has to the second.

If, among the equimultiples of four magnitudes, compared as in the fifth definition, we should find ●●●●● ⊏ □□□□, but ▲▲▲▲▲ = or ⊐ △△△△, or if we should find any particular multiple M′ of the first and third, and a particular multiple *m*′ of the second and fourth, such, that M′ times the first is ⊏ *m*′ times the second, but M′ times the third is not ⊏ *m*′ times the fourth, *i.e.*, = or ⊐ *m*′ times the fourth; then, the first is said to have to the second a greater ratio than the third has to the fourth; or the third has to the fourth, under such circumstances, a less ratio than the first has to the second: although several other equimultiples may tend to show that the four magnitudes are proportionals.

This definition will in future be expressed thus:—

If M′ ⬠1 ⊏ *m*′ ◡2, but M′ □3 = or ⊐ *m*′ ◇4,

then ⬠1 : ◡2 ⊏ □3 : ◇4.

In the above general expression, M′ and *m*′ are to be considered particular multiples, not like the multiples M and *m* introduced in the fifth definition, which are in that definition considered to be every pair of multiples that can be taken. It must also be here observed, that ⬠1, ◡2, □3, and the like symbols, are to be considered merely the representatives of geometrical magnitudes.

In a partial arithmetical way, this may be set forth as follows:—

Let us take the four numbers, 8, 7, 10, and 9.

First. 8	Second. 7	Third. 10	Fourth. 9
16	14	20	18
24	21	30	27
32	28	40	36
40	35	50	45
48	42	60	54
56	49	70	63
64	56	80	72
72	63	90	81
80	70	100	90
88	77	110	99
96	84	120	108
104	91	130	117
112	98	140	126
&c.	&c.	&c.	&c.

Among the above multiples we find 16 ⊏ 14 and 20 ⊏ 18; that is, twice the first is greater than twice the second, and twice the third is greater than twice the fourth; and 16 ⊐ 21 and 20 ⊐ 27; that is, twice the first is less than three times the second, and twice the third is less than three times the fourth; and among the same multiples we can find 72 ⊏ 56 and 90 ⊏ 72: that is, 9 times the first is greater than 8 times the second, and 9 times the third is greater than 8 times the fourth. Many other equimultiples might be selected, which would tend to show that the numbers 8, 7, 10, 9, were proportionals, but they are not, for we can find a multiple of the first ⊏ a multiple of the second, but the same multiple of the third that has been taken of the first not ⊏ the same multiple of the fourth which has been taken of the second; for instance, 9 times the first is ⊏ 10 times the second, but 9 times the third is not ⊏ 10 times the fourth, that is, 72 ⊏ 70, but 90 not ⊏ 90, or 10 times the first we find ⊏ 11 times the second, but 10 times the third not greater than 11 times the fourth. When any such multiples as these can be found, the first (8) is said to have to the second (7) a greater ratio than the third (10) has to the fourth (9).

PROP. VIII. THEO.

Of unequal magnitudes the greater has a greater ratio to the same than the less has: and the same magnitude has a greater ratio to the less than it has to the greater.

Let $\overset{\triangle}{\square}$ and □ be two unequal magnitudes,
and ○ any other.

We shall first prove that $\overset{\triangle}{\square}$, which is the greater of the two unequal magnitudes, has a greater ratio to ○ than □, the less, has to ○;

that is, $\overset{\triangle}{\square}$: ○ ⊏ □ : ○.
take M′ $\overset{\triangle}{\square}$, *m′* ○, M′ □, and *m′* ○;
such, that M′ △ and M′ □ shall be each ⊏ ○;
also take *m′* ○ the least multiple of ○,
which will make *m′* ○ ⊏ M′ □;
∴ M′ □ is not ⊏ *m′* ○,
but M′ $\overset{\triangle}{\square}$ is ⊏ *m′* ○, for,
as *m′* ○ is the first multiple which first becomes ⊏ M′ □,
than (*m′* — 1) ○ or *m′* ○ — ○ is not ⊏ M′ □,
and ○ is ⊏ M′ △,

∴ m′○ — ○ + ○ must be ⊐ M′□ + M′△;

that is, m′○ must be ⊐ M′$\overset{\triangle}{\square}$;

∴ M′$\overset{\triangle}{\square}$ is ⊏ m′○; but it has been shown above that M′□ is not ⊏ m′○, therefore, by the seventh definition,

$\overset{\triangle}{\square}$ has to ○ a greater ratio than □ : ○.

Next we shall prove that ○ has a greater ratio to □, the less, than it has to $\overset{\triangle}{\square}$, the greater;

or, ○ : □ ⊏ ○ : $\overset{\triangle}{\square}$.

Take m′○, M′□, m′○, and M′$\overset{\triangle}{\square}$.

the same as in the first case, such, that

M′△ and M′□ will be each ⊏ ○, and m′○ the least multiple of ○, which first becomes greater than M′□.

∴ m′○ — ○ is not ⊏ M′□,

and ○ is not ⊐ M′△; consequently

m′○ — ○ + ○ is ⊐ M′□ + M′△;

∴ m′○ is ⊐ M′$\overset{\triangle}{\square}$, and ∴ by the seventh definition,

○ has to □ a greater ration than ○ has to $\overset{\triangle}{\square}$.

∴ Of unequal magnitudes, &c.

The contrivance employed in this proposition for finding, among the multiples taken, as in the fifth definition, a multiple of the first greater than the multiple of the second, but the same multiple of the third which has been taken of the first, not greater than the same multiple of the fourth which has been taken of the second, may be illustrated numerically as follows:—

The number 9 has a greater ratio to 7 than 8 has to 7; that is, $9 : 7 \sqsubset 8 : 7$; or, $8 + 1 : 7 \sqsubset 8 : 7$.

The multiple of 1, which first becomes greater than 7, is 8 times, therefore we may multiply the first and third by 8, 9, 10, or any other greater number; in this case, let us multiply the first and third by 8, and we have 64 + 8 and 64: again, the first multiple of 7 which becomes greater than 64 is 10 times; then, by multiplying the second and fourth by 10, we shall have 70 and 70; then, arranging these multiples, we have—

8 times the first.	10 times the second.	8 times the third.	10 times the fourth.
64 + 8	70	64	70

Consequently 64 + 8, or 72, is greater than 70, but 64 is not greater than 70, ∴ by the seventh definition, 9 has a greater ratio to 7 than 8 has to 7.

The above is merely illustrative of the foregoing demonstration, for this property could be shown of these or other numbers very readily in the following manner; because, if an antecedent contains its consequent a greater number of times than another antecedent contains its consequent, or when a fraction is formed of an antecedent for the numerator, and its consequent for the denominator be greater than another fraction which is formed of another antecedent for the numerator and its consequent for the denominator, the ratio of the first antecedent to its consequent is greater than the ratio of the last antecedent to its consequent.

Thus, the number 9 has a greater ratio to 7, than 8 has to 7, for $\frac{9}{7}$ is greater than $\frac{8}{7}$.

Again, 17 : 19 is a greater ratio than 13 : 15, because $\frac{17}{19} = \frac{17 \times 15}{19 \times 15} = \frac{255}{285}$, and $\frac{13}{15} = \frac{13 \times 19}{15 \times 19} = \frac{247}{285}$, hence it is evident that $\frac{255}{285}$ is greater than $\frac{247}{285}$, $\therefore$ $\frac{17}{19}$ is greater than $\frac{13}{15}$, and, according to what has been above shown, 17 has to 19 a greater ratio than 13 has to 15.

So that the general terms upon which a greater, equal, or less, ratio exists are as follows :—

If $\frac{A}{B}$ be greater than $\frac{C}{D}$, A is said to have to B a greater ratio than C has to D; if $\frac{A}{B}$ be equal to $\frac{C}{D}$, then A has to B the same ratio which C has to D; and if $\frac{A}{B}$ be less than $\frac{C}{D}$, A is said to have to B a less ratio than C has to D.

The student should understand all up to this proposition perfectly before proceeding further, in order fully to comprehend the following propositions of this book. We therefore strongly recommend the learner to commence again, and read up to this slowly, and carefully reason at each step, as he proceeds, particularly guarding against the mischievous system of depending wholly on the memory. By following these instructions, he will find that the parts which usually present considerable difficulties will present no difficulties whatever, in prosecuting the study of this important book.

PROP. IX. THEO.

Magnitudes which have the same ratio to the same magnitudes are equal to one another; and those to which the same magnitude has the same ratio are equal to one another.

Let ◇ : □ :: ○ : □,

then ◇ = ○.

For, if not, let ◇ ⊏ ○, then will

◇ : □ ⊏ ○ : □ (8 B. v.),

which is absurd according to the hypothesis.

∴ ◇ is not ⊏ ○.

In the same manner it may be shown, that

○ is not ⊏ ◇,

∴ ◇ = ○.

Again, let □ : ◇ :: □ : ○, then will

◇ = ○.

For (invert.) ◇ : □ :: ○ : □,

therefore, by the first case, ◇ = ○.

∴ Magnitudes which have the same ratio, &c.

This may be shown otherwise, as follows:—

Let A : B = A : C, then B = C, for, as the fraction $\frac{A}{B}$ = the fraction $\frac{A}{C}$, and the numerator of one equal to the numerator of the other, therefore the denominator of these fractions are equal, that is, B = C.

Again, if B : A = C : A, B = C. For, as $\frac{B}{A} = \frac{C}{A}$, B must = C.

PROP. X. THEO.

That magnitude which has a greater ratio than another has unto the same magnitude, is the greater of the two; and that magnitude to which the same has a greater ratio than it has unto another magnitude, is the less of the two.

Let ▽ : □ ⊏ ○ : □, then ▽ ⊏ ○.

For if not, let ▽ = or ⊐ ○;

then, ▽ : □ = ○ : □ (7 B. v.) or

▽ : □ ⊐ ○ : □ (8 B v.) and (invert.),

which is absurd according to the hypothesis.

∴ ▽ is not = or ⊐ ○, and

∴ ▽ must be ⊏ ○.

Again, let □ : ○ ⊏ □ : ▽,

then, ○ ⊐ ▽.

For if not, ○ must be ⊏ or = ▽,

then □ : ○ ⊐ □ : ▽ (8 B. v.) and (invert.),

or □ : ○ = □ : ▽ (7 B. v.), which is absurd (hyp.);

∴ ○ is not ⊏ or = ▽,

and ∵ ○ must be ⊐ ▽.

∴ That magnitude which has, &c.

The truth of this proposition may be shown thus :—

Let $A : B \mathrel{-\!\!\sqsupset} C : B$, then, $A \mathrel{\sqsubset\!\!-} C$, for as $\frac{A}{B} \mathrel{\sqsubset\!\!-} \frac{C}{B}$, A must be $\mathrel{\sqsubset\!\!-}$ C, by the reasoning before employed. And if $B : C \mathrel{\sqsubset\!\!-} B : A$, $C \mathrel{-\!\!\sqsupset} A$, for the fraction $\frac{B}{C} \mathrel{\sqsubset\!\!-} \frac{B}{A}$, and that this may be the case, C must be less than A.

PROP. XI. THEO.

Ratios that are the same to the same ratio are the same to each other.

Let ◇ : □ = ○ : ⬠ and ○ : ⬠ = ▲ : △,

then will ◇ : □ = ▲ : △.

For if M ◇ ⊏, =, or ⊐ *m* □, then M ○ ⊏, =, or ⊏ *m* ⬠

and if M ○ ⊏, =, or ⊐ *m* ⬠, then M ▲ ⊏, =, or ⊐ *m* △,

(5 def. v.);

∴ if M ◇ ⊏, =, or ⊐ *m* □, M ▲ ⊏, =, or ⊐ *m* △,

and ∴ (5 def. v.) ◇ : □ = ▲ : △.

∴ Ratios that are the same, &c.

Let A : B = C : D

and A : B = E : F,

then C : D = E : F, because

$$\frac{A}{B} = \frac{C}{D}$$

$$\text{and } \frac{A}{B} = \frac{E}{F}$$

$$\therefore \frac{C}{D} = \frac{E}{F}$$

and ∴ C : D = E : F.

PROP. XII. THEO.

If any number of magnitudes be proportionals, as one of the antecedents is to its consequent, so shall all the antecedents taken together be to all the consequents.

Let □ : ◯ = ◇ : ◊ = ⊔ : ▽ = ○ : △ = ▲ : ●;

then will

□ : ◯ = □ + ◇ + ⊔ + ○ + ▲ : ◯ + ◊ + ▽ + △ + ●.

For if M □ ⊏ m ◯, then M ◇ ⊏ m ◊,

and M ⊔ ⊏ m ▽

M ○ ⊏ m △,

also M ▲ ⊏ m ●. (5 def. v.)

Therefore, if M □ ⊏ m ◯, then will

M □ + M ◇ + M ⊔ + M ○ + M ▲,

or M (□ + ◇ + ⊔ + ○ + ▲)

be greater than m ◯ + m ◊ + m ▽ + m △ + m ●,

or m (◯ + ◊ + ▽ + ▲ + ●).

In the same way it may be shown, if M times one of the antecedents be equal to or less than m times one of the consequents, M times all the antecedents taken together, will be equal to or less than m times all the consequents taken together. Therefore, by the fifth definition, as one of the antecedents is to its consequent, so is all the antecedents taken together to all the consequents taken together.

$\therefore$ If any number of magnitudes, &c.

Arithmetical Illustration.

$$12 : 16 = 24 : 32 = 6 : 8 = 3 : 4.$$

Antecedents.	Consequents.
12	16
24	32
6	8
3	4
45	60

$$12 : 16 :: 45 : 60, \text{ or } 24 : 32 = 45 : 60, \&c.$$

Algebraical Exposition.

Let $a : b = c : d = e : f = g : h$

$$\frac{a}{b} = \frac{a}{b} \quad \therefore ab = ab$$

$$\frac{a}{b} = \frac{c}{d} \quad \therefore ad = bc$$

$$\frac{a}{b} = \frac{e}{f} \quad \therefore af = be$$

$$\frac{a}{b} = \frac{g}{h} \quad \therefore ah = bg,$$

and therefore $ab + ad + af + ah = ba + bc + be + bg$,

or $a(b + d + f + h) = b(a + c + e + g)$,

$$\therefore a : b = a + c + e + g : b + d + f + h.$$

PROP. XIII. THEO.

If the first has to the second the same ratio which the third has to the fourth, but the third to the fourth a greater ratio than the fifth has to the sixth; the first shall also have to the second a greater ratio than the fifth to the sixth.

Let $1 : 2 = 3 : 4$, but $3 : 4 \sqsubset 5 : 6$,

then $1 : 2 \sqsubset 5 : 6$.

For, because $3 : 4 \sqsubset 5 : 6$, there are some multiples

(M′ and *m*′) of 3 and 5, and of 4 and 6,

such that $M' \, 3 \sqsubset m' \, 4$,

but $M' \, 5$ not $\sqsubset m' \, 6$, by the seventh definition.

Let these multiples be taken, and take the same multiples of

1 and 2.

∴ (5 def. v.) if $M' \, 1 \sqsubset, =,$ or $\sqsupset m' \, 2$;

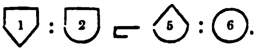
then will $M' \, 3 \sqsubset, =,$ or $\sqsupset m' \, 4$,

but $M' \, 3 \sqsubset m' \, 4$ (construction);

∴ $M' \, 1 \sqsubset m' \, 2$,

but $M' \, 5$ is not $\sqsubset m' \, 6$ (construction);

and therefore by the seventh definition,



∴. If the first has to the second, &c.

$$\text{Let } A : B = C : D, \text{ but } C : D \sqsubset E : F,$$

$$\text{then } A : B \sqsubset E : F.$$

$$\text{For } \frac{A}{B} = \frac{C}{D}, \text{ but } \frac{C}{D} \sqsubset \frac{E}{F},$$

$$\therefore \frac{A}{B} \sqsubset \frac{E}{F} \text{ and } \therefore A : B \sqsubset E : F.$$

PROP. XIV. THEO.

If the first has the same ratio to the second which the third has to the fourth; then, if the first be greater than the third, the second shall be greater than the fourth; and if equal, equal; and if less, less.

Let 1 : 2 :: 3 : 4, and first suppose
1 ⊏ 3, then will 2 ⊏ 4.
For 1 : 2 ⊏ 3 : 2 (8 B. v.), and by the hypothesis,
1 : 2 = 3 : 4;
∴ 3 : 4 ⊏ 3 : 2 (13 B. v.),
∴ 4 ⊐ 2 (10 B. v.),
or 2 ⊏ 4.

Secondly, let 1 = 3, then will 2 = 4.
For 1 : 2 = 3 : 2 (7 B. v.),
and 1 : 2 = 3 : 4 (hyp.);
∴ 3 : 2 = 3 : 4 (11 B v.),
and ∴ 2 = 4 (9 B v.).

Thirdly, if $\boxed{1} \sqsupset \boxed{3}$, then will $\boxed{2} \sqsupset \boxed{4}$;

because $\boxed{3} \sqsubset \boxed{1}$ and

$\boxed{3} : \boxed{4} = \boxed{1} : \boxed{2}$;

$\therefore$ $\boxed{4} \sqsubset \boxed{2}$, by the first case,

that is, $\boxed{2} \sqsupset \boxed{4}$.

$\therefore$ If the first has the same ratio, &c.

Let $A : B = C : D$.

If $A \sqsubset, =,$ or $\sqsupset C$, then will $B \sqsubset, =,$ or $\sqsupset D$.

First, let $A \sqsubset C$,

then it is evident $\frac{A}{B} \sqsubset \frac{C}{B}$,

but $\frac{A}{B} = \frac{C}{D}$;

$\therefore \frac{C}{D} \sqsubset \frac{C}{B}$;

$\therefore$ $CB \sqsubset CD$,

and $\therefore$ $B \sqsubset D$.

In the same way it may be readily shown,

if A be $=$ or $\sqsupset C$, B will be $=$ or $\sqsupset D$.

PROP. XV. THEO.

Magnitudes have the same ratio to one another which their equimultiples have.

Let ○ and □ be two magnitudes;

then, ○ : □ :: M′ ○ : M′ □.

For ○ : □ = ○ : □

= ○ : □

= ○ : □

∴ ○ : □ :: 4 ○ : 4 □. (12 B. v.).

And as the same reasoning is generally applicable, we have

○ : □ :: M′ ○ : M′ □.

∴ Magnitudes have the same ratio, &c.

Let A and B be two numbers, and C and D equimultiples of them,

$$A : B :: C : D.$$

Let $C = n\,A$ and $D = n\,B$;

then it is evident that $A : B :: n\,A : n\,B$,

for $\frac{A}{B} = \frac{n\,A}{n\,B}$.

DEFINITION XIII.

The technical term permutando, or alternando, by permutation or alternately, is used when there are four proportionals, and it is inferred that the first has the same ratio to the third which the second has to the fourth; or that the first is to the third as the second is to the fourth: as is shown in the following proposition:—

Let ● : ▲ :: ○ : △,

by "permuntando" or "alternando" it is inferred ● : ○ :: ▲ : △.

It may be necessary here to remark that the magnitudes ●, ▲, ○, △, must be homogeneous, that is, of the same nature or similitude of kind; we must therefore, in such cases, compare lines with lines, surfaces with surfaces, solids with solids, &c. Hence the student will readily perceive that a line and a surface, a surface and a solid, or other heterogenous magnitudes, can never stand in the relation of antecedent and consequent.

PROP. XVI. THEO.

If four magnitudes of the same kind be proportionals, they are also proportionals when taken alternately.

Let 1 : 2 :: 3 : 4, then 1 : 3 :: 2 : 4.

For M 1 : M 2 :: 1 : 2 (15 B. v.),

and M 1 : M 2 :: 3 : 4 (hyp.) and (11 B. v.);

Also *m* 3 : *m* 4 :: 3 : 4 (15 B. v.),

∴ M 1 : M 2 :: *m* 3 : *m* 4 (14 B. v.),

and ∴ if M 1 ⊏, =, or ⊐ *m* 3,

then will M 2 ⊏, =, or ⊐ *m* 4 (14 B. v.);

therefore, by the fifth definition,

1 : 3 :: 2 : 4.

∴ If four magnitudes of the same kind, &c.

If $a : b :: c : d$ then will $a : c :: b : d$.

Because the quantities are proportionals,

$$\frac{a}{b} = \frac{c}{d},$$

multiply each of these fractions by $\frac{b}{c}$, and we have

$$\frac{ab}{bc} = \frac{cb}{cd}, \text{ or } \frac{a}{c} = \frac{b}{d};$$

$$\therefore\ a : c :: b : d.$$

Unless the four quantities are of the same kind the alternation, as before observed, cannot take place, because the operation supposes the first to be some multiple, parts or part, of the third.

DEFINITION XVI.

Dividendo, by division, when there are four proportionals, and it is inferred that the excess of the first above the second is to the second as the excess of the third above the fourth is to the fourth.

Let $a : b :: c : d$;

by "dividendo" it is inferred $a - b : b :: c - d : d$.

According to the above, a is supposed to be greater than b, and c greater than d; if this be not the case, but to have b greater than a, and d greater than c, b and d can be made to stand as antecedents, and a and c as consequents, by "invertion"

$$b : a :: d : c;$$

then, by "dividendo," we infer $b - a : a :: d - c : c$.

PROP. XVII. THEO.

If magnitudes, taken jointly, be proportionals, they shall also be proportionals when taken separately: that is, if two magnitudes together have to one of them the same ratio which two others have to one of these, the remaining one of the first two shall have to the other the same ratio which the remaining one of the last two has to the other of these.

Let $\boxed{1} + \boxed{2} : \boxed{2} :: \boxed{3} + \boxed{4} : \boxed{4}$,

then will $\boxed{1} : \boxed{2} :: \boxed{3} : \boxed{4}$.

Take $M\boxed{1} \sqsubset m\boxed{2}$, to each add $M\boxed{2}$,

then we have $M\boxed{1} + M\boxed{2} \sqsubset m\boxed{2} + M\boxed{2}$,

or $M(\boxed{1} + \boxed{2}) \sqsubset (m + M)\boxed{2}$:

but because $\boxed{1} + \boxed{2} :: \boxed{2} :: \boxed{3} + \boxed{4} : \boxed{4}$ (hyp.),

and $M(\boxed{1} + \boxed{2}) \sqsubset (m + M)\boxed{2}$;

$\therefore M(\boxed{3} + \boxed{4}) \sqsubset (m + M)\boxed{4}$ (5 def. v.);

$\therefore M\boxed{3} + M\boxed{4} \sqsubset m\boxed{4} + M\boxed{4}$;

$\therefore M\boxed{3} \sqsupset m\boxed{4}$, by taking $M\boxed{4}$ from both sides:

that is, when $M\boxed{1} \sqsubset m\boxed{2}$, then $M\boxed{3} \sqsubset m\boxed{4}$.

In the same manner it may be proved,

that if M $\boxed{1}$ $=$ or $\sqsupset$ m $\boxed{2}$, then will M $\boxed{3}$ $=$ or $\sqsupset$ m $\boxed{4}$;

and ∴ $\boxed{1} : \boxed{2} :: \boxed{3} : \boxed{4}$. (5 def v.).

∴ If magnitudes taken jointly, &c.

Arithmetical Illustration.

Let us take the proposition

$7 : 4 :: 21 : 12$,

or $3 + 4 : 4 :: 12 + 9 : 12$;

then $3 : 4 :: 9 : 12$.

Algebraical Exposition.

Let $a + b : b :: c + d : d$,

then $a : b :: c : d$.

Because $a + b : b :: c + d : d$;

$$\therefore \frac{a+b}{b} = \frac{c+d}{d},$$

$$\therefore \frac{a}{b} + 1 = \frac{c}{d} + 1,$$

$$\therefore \frac{a}{b} = \frac{c}{d},$$

and ∴ $a : b :: c : d$.

DEFINITION XV.

The term componendo, by composition, is used when there are four proportionals; and it is inferred that the first together with the second is to the second as the third together with the fourth is to the fourth.

Let $A : B :: C : D$;

then, by the term "componendo," it is inferred that

$A + B : B :: C + D : D$.

By "invertion" B and D may become the first and third, A and C the second and fourth, as

$B : A :: D : C$,

then, by "componendo," we infer that

$B + A : A :: D + C : C$.

PROP. XVIII. THEO.

If magnitudes, taken separately, be proportionals, they shall also be proportionals when taken jointly: that is, if the first be to the second as the third is to the fourth, the first and second together shall be to the second as the third and fourth together is to the fourth.

Let ▽1 : ◡2 :: □3 : ◇4,

then ▽1 + ◡2 : ◡2 :: □3 + ◇4 : ◇4;

for if not, let ▽1 + ◡2 : ◡2 :: □3 + ○ : ○,

supposing ○ not = ◇4;

∴ ▽1 : ◡2 :: □3 : ○ (17 B. v.);

but ▽1 : ◡2 :: □3 : ◇4 (hyp.);

∴ □3 : ○ :: □3 : ◇4 (11 B. v.);

∴ ○ = ◇4 (9 B. v.),

which is contrary to the supposition;

∴ ○ is not unequal to ◇4,

that is ○ = ◇4;

∴ ▽1 + ◡2 : ◡2 :: □3 + ◇4 : ◇4.

∴ If magnitudes, taken separately, &c.

This proposition may be illustrated arithmetically as follows:—

Let us take the proportion

$$4 : 5 :: 16 : 20,$$

then $4 + 5 : 5 :: 16 + 20 : 20$;

or $9 : 5 :: 36 : 20$.

Algebraical Exposition.

Let $a : b :: c : d$,

then $a + b : b :: c + d : d$;

because $a : b : c : d$,

$$\frac{a}{b} = \frac{c}{d},$$

to each of these fractions add 1,

$$\therefore \frac{a}{b} + 1 = \frac{c}{d} + 1,$$

$$\therefore \frac{a+b}{b} = \frac{c+d}{d},$$

and $\therefore a + b : b :: c + d : d$.

It may also be very readily shown that $a + b : a :: c + d : c$.

For if $\frac{a}{b} = \frac{c}{d}$, and $\frac{b}{a} = \frac{d}{c}$,

then will $\frac{b}{a} + 1 = \frac{d}{c} + 1$;

$$\therefore \frac{b+a}{a} = \frac{d+c}{c},$$

and consequently $a + b : a :: d + c : c$.

PROP. XIX. THEO.

If a whole magnitude be to a whole, as a magnitude taken from the first, is to a magnitude taken from the other; the remainder shall be to the remainder, as the whole to the whole.

Let $\bigtriangledown_{1} + \cup_{2} : \square_{3} + \Diamond_{4} :: \bigtriangledown_{1} : \square_{3}$,

then will $\cup_{2} : \Diamond_{4} :: \bigtriangledown_{1} + \cup_{2} : \square_{3} + \Diamond_{4}$,

For $\bigtriangledown_{1} + \cup_{2} : \bigtriangledown_{1} :: \square_{3} + \Diamond_{4} : \square_{3}$ (alter.),

$\therefore$ $\cup_{2} : \bigtriangledown_{1} :: \Diamond_{4} : \square_{3}$ (divid.),

again $\cup_{2} : \Diamond_{4} :: \bigtriangledown_{1} : \square_{3}$ (alter.),

but $\bigtriangledown_{1} + \cup_{2} : \square_{3} + \Diamond_{4} :: \bigtriangledown_{1} : \square_{3}$ (hyp.);

therefore $\cup_{2} : \Diamond_{4} :: \bigtriangledown_{1} + \cup_{2} : \square_{3} + \Diamond_{4}$ (11 B. v.).

$\therefore$ If a whole magnitude be to a whole, &c.

Arithmetical Illustration.

Take the proportion

21 : 9 :: 7 : 3,

14 + 7 : 6 + 3 :: 7 : 3;

then 14 : 6 :: 14 + 7 : 6 + 3,

or 14 : 6 : 21 : 9.

Algebraical Exposition.

Let $a + b : c + d :: b : d$, then $a : c :: a + b : c + d$,

because $a + b : c + d :: b : d$,

$$\therefore\ a + b : b :: c + d : d \text{ (alter.)},$$

$$\therefore\ \frac{a+b}{b} = \frac{c+d}{d},$$

$$\therefore\ \frac{a}{b} + 1 = \frac{c}{d} + 1,$$

$$\therefore\ \frac{a}{b} = \frac{c}{d},$$

$$\therefore\ a : b :: c : d,$$

$$\therefore\ a : c :: b : d;$$

but, by the hypothesis, $a + b : c + d :: b : d$,

and $\therefore\ a : c :: a + b : c + d$.

Cor. If the whole be to the whole, as a magnitude taken from the first, is to a magnitude taken from the other; the remainder likewise is to the remainder, as the magnitude taken from the first to that taken from the other. The demonstration is contained in that of the preceding proposition

DEFINITION XVII.

The term "convertendo," by conversion, is made use of by geometricians, when there are four proportionals, and it is inferred, that the first is to its excess above the second, as the third is to its excess above the fourth. See the following proposition:—

PROP. E. THEO.

If four magnitudes be proportionals, they are also proportionals by conversion: that is, the first is to its excess above the second, as the third to its excess above the fourth.

Let ○△ : △ :: □◇ : ◇,

then shall ○△ : ○ :: □◇ : □.

Because ○△ : △ : □◇ : ◇;

therefore ○ : △ :: □ : ◇ (divi.),

∴ △ : ○ :: ◇ : □ (inver.),

∴ △○ : ○ :: ◇□ : □ (compo.).

∴ If four magnitudes, &c.

Arithmetical Illustration.

Let the four numbers that are proportionals be
17, 3, 51, and 9;
then 17 : (17 − 3) :: 51 : (51 − 9),
or 17 : 14 :: 51 : 42.

Algebraical Exposition

Let $a : b :: c : d$,

where a is to be considered greater than b, and c greater than d;

then shall $a : a - b :: c : c - d$;

because $a : b :: c : d$,

$\therefore\ b : a :: d : c$,

$\therefore\ a - b : a :: c - d : c$,

$\therefore\ a : a - b :: c : c - d$.

DEFINITION XVIII.

"Ex æquali" (sc. distantiâ), or ex æquo, from equality of distance : when there is any number of magnitudes more than two, and as many others, such that they are proportionals when taken two and two of each rank, and it is inferred that the first is to the last of the first rank of magnitudes, as the first is to the last of the others: "of this there are the two following kinds, which arise from the different order in which the magnitudes are taken, two and two."

DEFINITION XIX

"Ex æquali," from equality. This term is used simply by itself, when the first magnitude is to the second of the first rank, as the first to the second of the other rank ; and as the second is to the third of the first rank, so is the second to the third of the other ; and so on in order: and the inference is as mentioned in the preceding definition ; whence this is called ordinate proportion. It is demonstrated in the 22nd Prop., Book v.

Thus, if there be two ranks of magnitudes,

A, B, C, D, E, F, the first rank,
and L, M, N, O, P, Q, the second,
such that A : B :: L : M, B : C :: M : N,
C : D :: N : O, D : E :: O : P, E : F :: P : Q;
we infer by the term "ex æquali" that
A : F :: L : Q.

DEFINITION XX.

"Ex æquali in proportione perturbatâ seu inordinatâ," from equality in perturbate, or disorderly proportion. This term is used when the first magnitude is to the second of the first rank as the last but one is to the last of the second rank; and as the second is to the third of the first rank, so is the last but two to the last but one of the second rank; and as the third is to the fourth of the first rank, so is the third from the last to the last but two of the second rank; and so on in a cross order: and the inference is in the 18th definition. It is demonstrated in 23 Prop., Book v.

Thus, if there be two ranks of magnitudes,

A, B, C, D, E, F, the first rank,
and L, M, N, O, P, Q, the second,
such that A : B :: P : Q, B : C :: O : P,
C : D :: N : O, D : E :: M : N, E : F :: L : M;
the term "ex æquali in proportione perturbatâ seu inordinatâ" infers that
A : F :: L : Q.

PROP. XX. THEO.

If there be three magnitudes, and other three, which, taken two and two, have the same ratio; then, if the first be greater than the third, the fourth shall be greater than the sixth; and if equal, equal; and if less, less.

Let ▽, ⊔, □, be the first three magnitudes,

and ◇, ⬠, ○, be the other three,

such that ▽ : ⊔ :: ◇ : ⬠, and ⊔ : □ :: ⬠ : ○.

Then, if ▽ ⊏, =, or ⊐ □, then will ◇ ⊏, =, or ⊐ ○.

From the hypothesis, by alternando, we have

▽ : ◇ :: ⊔ : ⬠,

and ⊔ : ⬠ :: □ : ○;

∴ ▽ : ◇ :: □ : ○ (11 B. v.);

∴ if ▽ ⊏, =, or ⊐ □, then will ◇ ⊏, =, or ⊐ ○

(14 B. v.)

∴. If there be three magnitudes, &c.

PROP. XXI. THEO.

If there be three magnitudes, and other three which have the same ratio, taken two and two, but in a cross order; then if the first magnitude be greater than the third, the fourth shall be greater than the sixth; and if equal, equal; and if less, less.

Let ▽, ⊔, □, be the first three magnitudes,

and ◇, ⬠, ○, the other three,

such that ▽ : ⊔ :: ⬠ : ○, and ⊔ : □ :: ◇ : ⬠.

Then, if ▽ ⊏, =, or ⊐ □, then will ◇ ⊏, =, or ⊐ ○.

First, let ▽ be ⊏ □:

then, because ⊔ is any other magnitude,

▽ : ⊔ ⊏ □ : ⊔ (8 B. v.);

but ⬠ : ○ :: ▽ : ⊔ (hyp.);

∴ ⬠ : ○ ⊏ □ : ⊔ (13 B. v.);

and because ⊔ : □ :: ◇ : ⬠ (hyp.),

∴ □ : ⊔ :: ⬠ : ◇ (inv.),

and it was shown that ⬠ : ○ ⊏ □ : ⊔,

∴ ◊ : ○ ⊏ ◊ : ◇ (Cor. 13, B. v.),

∴ ○ ⊐ ◇,

that is ◇ ⊏ ○.

Secondly, let ▽ = □ ; then shall ◇ = ○.

For because ▽ = □,

▽ : ⊔ = □ : ⊔ (7 B. v.);

but ▽ : ⊔ = ◊ : ○ (hyp.),

and □ : ⊔ = ◊ : ◇ (hyp. and inv.),

∴ ◊ : ○ = ◊ : ◇ (11 B. v.),

∴ ◇ = ○ (9 B. v.)

Next, let ▽ be ⊐ □, then ◇ shall be ⊐ ○;

for □ ⊏ ▽,

and it has been shown that □ : ⊔ = ◊ : ◇,

and ⊔ : ▽ = ○ : ◊;

∴ by the first case ○ is ⊏ ◇,

that is, ◇ ⊐ ○.

∴ If there be three, &c.

PROP. XXII. THEO.

If there be any number of magnitudes, and as many others, which, taken two and two in order, have the same ratio; the first shall have to the last of the first magnitudes the same ratio which the first of the others has to the last of the same.

N.B.—This is usually cited by the words "ex æquali," or "ex æquo."

First, let there be magnitudes ▽, ⊔, □,

and as many others ◇, △, ○,

such that

▽ : ⊔ :: ◇ : △,

and ⊔ : □ :: △ : ○;

then shall ▽ : □ :: ◇ : ○.

Let these magnitudes, as well as any equimultiples whatever of the antecedents and consequents of the ratios, stand as follows:—

▽, ⊔, □, ◇, △, ○,

and

M ▽, *m* ⊔, N □, M ◇, *m* △, N ○,

because ▽ : ⊔ :: ◇ : △;

∴ M ▽ : *m* ⊔ :: M ◇ : *m* △ (4 B. v.).

For the same reason

m ∪ : N □ :: *m* ⬠ : N ○;

and because there are three magnitudes,

M ▽, *m* ∪, N □,

and other three, M ◇, *m* ⬠, N ○,

which, taken two and two, have the same ratio ;

∴ if M ▽ ⊏, =, or ⊐ N □,

then will M ◇ ⊏, =, or ⊐ N ○, by (20 B. v.),

and ∴ ▽ : □ :: ◇ : ○ (def. 5).

Next, let there be four magnitudes, ▽1, ∪2, □3, ◇4,

and other four, ⬠1, ○2, ⊓3, ∧4,

which, taken two and two, have the same ratio, that is to say,

▽1 : ∪2 :: ⬠1 : ○2,

∪2 : □3 :: ○2 : ⊓3,

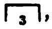
and □3 : ◇4 :: ⊓3 : ∧4,

then shall ▽1 : ◇4 :: ⬠1 : ∧4;

for, because ▽1, ∪2, □3, are three magnitudes,

and ⬠1, ○2, ⊓3, other three,

which, taken two and two, have the same ratio;

therefore, by the foregoing case, : :: : ,

but : :: : ;

therefore again, by the first case, : :: : ;

and so on, whatever the number of magnitudes be.

∴ If there be any number, &c.

Arithmetical Illustration.

Let there be two ranks of numbers,
3, 4, 2, 1, 5, the first rank,
and 9, 12, 6, 3, 15, the second;
such that 3 : 4 :: 9 : 12
4 : 2 :: 12 : 6
2 : 1 :: 6 : 3
1 : 5 :: 3 : 15
then shall 3 : 5 :: 9 : 15.

Algebraical Exposition.

Let a, b, c, d, e, f, g, represent the first rank,
and a', b', c', d', e', f', g', the second,
such that $a : b :: a' : b'$
$b : c :: b' : c'$
$c : d :: c' : d'$
$d : e :: d' : e'$
&c., &c.,
then shall $a : g :: a' : g'$.

$$\text{For } \frac{a}{b} = \frac{a'}{b'}$$

$$\frac{b}{c} = \frac{b'}{c'}$$

$$\frac{c}{d} = \frac{c'}{d'}$$

$$\frac{d}{e} = \frac{d'}{e'}$$

$$\frac{e}{f} = \frac{e'}{f'}$$

$$\frac{f}{g} = \frac{f'}{g'}$$

$$\therefore \frac{a}{b} \times \frac{b}{c} \times \frac{c}{d} \times \frac{d}{e} \times \frac{e}{f} \times \frac{f}{g} = \frac{a'}{b'} \times \frac{b'}{c'} \times \frac{c'}{d'} \times \frac{d'}{e'} \times \frac{e'}{f'} \times \frac{f'}{g'},$$

$$\text{or } \frac{a\,b\,c\,d\,e\,f}{b\,c\,d\,e\,f\,g} = \frac{a'\,b'\,c'\,d'\,e'\,f'}{b'\,c'\,d'\,e'\,f'\,g'}, \text{ or } \frac{a}{g} = \frac{a'}{g'},$$

$$\therefore a : g :: a' : g'.$$

PROP. XXIII. THEO.

If there be any number of magnitudes, and as many others, which, taken two and two in a cross order, have the same ratio; the first shall have to the last of the first magnitudes the same ratio which the first of the others has to the last of the same.

N.B.—This is usually cited by the words "ex æquali in proportione perturbatâ;" or "ex æquo perturbato."

First, let there be three magnitudes, ▽1, ▢2, □3,

and other three, ◇1, △2, ○3,

which, taken two and two in a cross order, have the same ratio;

that is, ▽1 : ▢2 :: △2 : ○3,

and ▢ : □3 :: ◇1 : △2,

then shall ▽1 : □3 :: ◇1 : ○3

Let these magnitudes and their respective equimultiples be arranged as follows:—

▽1, ▢2, □3, ◇1, △2, ○3,

M ▽1, M ▢2, *m* □3, M ◇1, *m* △2, *m* ○3,

then ▽1 : ▢2 :: M ▽1 : M ▢2 (15 B. v.);

and for the same reason

△2 : ○3 :: m △2 : m ○3;

but ▽1 : ◡2 :: △2 : ○3 (hyp.),

∴ M ▽1 : M ◡2 :: m △2 : m ○3 (11 B. v.);

and because ◡2 : □3 :: ◇1 : △2 (hyp.),

∴ M ◡2 : m □3 :: M ◇1 : m △2 (4 B. v.);

then, because there are three magnitudes, M ▽1, M ◡2, m □3,

and other three, M ◇1, m △2, m ○3,

which, taken two and two in a cross order, have the same ratio;

therefore, if M ▽1 ⊏, =, or ⊐ m □3,

then will M ◇1 ⊏, =, or ⊏ m ○3 (21 B. v.),

and ∴ ▽1 : □3 :: ◇1 : ○3 (5 def. v.).

Next, let there be four magnitudes,

▽1, ◡2, □3, ◇4,

and other four, △1, ○2, ⊓3, ∧4,

which, when taken two and two in a cross order, have the same ratio; namely,

▽1 : ◡2 :: ⊓3 : ∧4,

◡2 : □3 :: ○2 : ⊓3,

and □3 : ◇4 :: △1 : ○2,

then shall ▽1 : ◇4 :: △1 : ∧4.

For, because ▽1, ⊔2, □3 are three magnitudes,

and ○2, ⊓3, ∧4, other three,

which, taken two and two in a cross order, have the same ratio,

therefore, by the first case, ▽1 □3 :: ○2 : ∧4,

but □3 : ◇4 :: △1 : ○2,

therefore again, by the first case, ▽1 : ◇4 :: △1 : ∧4;

and so on, whatever be the number of such magnitudes.

∴ If there be any number, &c.

Arithmetical Illustration.

Let there be two ranks of numbers,

2, 3, 4, 2, 6, 9, the first rank,

and 8, 12, 36, 18, 24, 36, the second;

such that 2 : 3 :: 24 : 36

3 : 4 :: 18 : 24

4 : 2 :: 36 : 18

2 : 6 :: 12 : 36

and 6 : 9 :: 8 : 12

then will 2 : 9 :: 8 : 36,

for $\frac{2}{9} = \frac{8}{36}$.

Algebraical Exposition.

Let $a, b, c, d, e, f,$ represent the first rank,

and $a', b', c', d', e', f',$ the second,

such that $a : b :: e' : f'$

$b : c :: d' : e'$

$c : d :: c' : d'$

$d : e :: b' : c'$

and $e : f :: a' : b'$,

then shall $a : f :: a' : f'$.

For $$\frac{a}{b} = \frac{e'}{f'}$$

$$\frac{b}{c} = \frac{d'}{e'}$$

$$\frac{c}{d} = \frac{c'}{d'}$$

$$\frac{d}{e} = \frac{b'}{c'}$$

$$\frac{e}{f} = \frac{a'}{b'}$$

and $\therefore$ $$\frac{a\,b\,c\,d\,e}{b\,c\,d\,e\,f} = \frac{e'\,d'\,c'\,b'\,a'}{f'\,e'\,d'\,c'\,b'},$$

$$\therefore \quad \frac{a}{f} = \frac{a'}{f'},$$

$\therefore \quad a : f :: a' : f'$.

PROP. XXIV. THEO.

If the first has to the second the same ratio which the third has to the fourth, and the fifth to the second the same which the sixth has to the fourth, the first and fifth together shall have to the second the same ratio which the third and sixth together have to the fourth.

First.	**Second.**	**Third.**	**Fourth.**
1	2	3	4
Fifth.		**Sixth.**	
5		6	

Let $1 : 2 :: 3 : 4$,

and $5 : 2 :: 6 : 4$,

then $1 + 5 : 2 :: 3 + 6 : 4$.

For $5 : 2 :: 6 : 4$ (hyp.),

and $2 : 1 :: 4 : 3$ (hyp.) and (inv.),

$\therefore 5 : 1 :: 6 : 3$ (22 B v.);

and, because these magnitudes are proportionals, they are proportionals when taken jointly,

$\therefore 1 + 5 : 5 :: 6 + 3 : 6$ (18 B. v.),

but $5 : 2 :: 6 : 4$ (hyp.),

$\therefore 1 + 5 : 2 :: 6 + 3 : 4$ (22 B. v.).

∴ If the first, &c.

Cor. 1. If the same hypothesis be made as in the proposition, the excess of the first and fifth shall be to the second as the excess of the third and sixth to the fourth. The demonstration of this is the same with that of the proposition, if division be used instead of composition.

Cor. 2. The proposition holds true of two ranks of magnitudes, whatever be their number, of which each of the first rank has to the second magnitude the same ratio that the corresponding one of the second rank has to a fourth magnitude; as is manifest.

Arithmetical Illustration

Let the number 6 (the first) have to 4 (the second) the same ratio which 3 (the third) has to 2 (the fourth), and 12 (the fifth) to 4 (the second) the same ratio which 6 (the sixth) has to 2 (the fourth).

First,	Second,	Third,	Fourth,	Fifth,	Sixth,
6	4	3	2	12	6

Then $12 + 6 : 4 :: 6 + 3 : 2$,

or $18 : 4 :: 9 : 2$.

Algebraical Exposition.

Let $a : b :: c : d$,

and $e : b :: f : d$;

then $a + e : b :: c + f : d$.

For $\frac{a}{b} = \frac{c}{d}$

and $\frac{e}{b} = \frac{f}{d}$

$\therefore \frac{a + e}{b} = \frac{c + f}{d}$, by addition.

and $\therefore a + e : b :: c + f : d$

PROP. XXV. THEO.

If four magnitudes of the same kind are proportionals, the greatest and least of them together are greater than the other two together.

Let four magnitudes, ▽ + ⊔, □ + ◇, ⊔, and ◇,

of the same kind, be proportionals, that is to say,

▽ + ⊔ : □ + ◇ :: ⊔ : ◇,

and let ▽ + ⊔ be the greatest of the four, and, consequently,

by proposition A and the 14th of this book, ◇ is the least;

then will ▽ + ⊔ + ◇ be ⊏ □ + ◇ + ⊔;

because ▽ + ⊔ : □ + ◇ :: ⊔ : ◇,

∴ ▽ : □ :: ▽ + ⊔ : □ + ◇ (19 B. v.),

but ▽ + ⊔ ⊏ □ + ◇ (hyp.),

∴ ▽ ⊏ □ (A. B. v.);

to each of these add ⊔ + ◇,

∴ ▽ + ⊔ + ◇ ⊏ □ + ⊔ + ◇.

∴ If four magnitudes, &c.

Arithmetical Illustration.

Let $2 : 3 :: 16 : 24$,

or $24 : 16 :: 3 : 2$;

here it is evident that $24 + 2 \sqsubset 16 + 3$;

the same is true of any other numbers that are proportionals.

Algebraical Exposition.

Let $a + m : b + n :: m : n$,

and let $a + m$ be the greatest,

then n will be the least, and $a + m + n \sqsubset b + n + n$.

Because $a + m : b + n :: m : n$,

$\therefore\ a : b :: a + m : b + n$;

but $a + m \sqsubset b + n$,

$\therefore\ a \sqsubset b$,

$\therefore\ a + n + m \sqsubset b + n + m$.

Note.—In the algebraic expositions, when $a : b :: c : d$, we immediately infer that $\frac{a}{b} = \frac{c}{d}$; and when $\frac{a}{b} = \frac{c}{d}$, we infer that $a : b :: c : d$.

These conclusions are drawn from considering, that when four magnitudes are proportionals, the first is the same multiple, parts, or part, of the second that the third is of the fourth; and that the fraction, which expresses the multiple, parts, or part, the first is of the second, or that the third is of the fourth, is a determinate fraction.

This fraction will be determinate when the two magnitudes which stand for antecedent and consequent have any common measure: for let m be a common measure of any two magnitudes, A and B.

Let m be contained in A, p times, $\therefore$ A $= m\,p$,
and m be contained in B, q times, $\therefore$ B $= m\,q$;

$$\text{and } \therefore \quad \frac{A}{B} = \frac{m\,p}{m\,q} = \frac{p}{q};$$

and as p and q are whole numbers,

the fraction $\frac{p}{q}$, or its equal, $\frac{A}{B}$, is determinate.

But it often happens that such magnitudes as A and B are incommensurable, or have no common measure, yet a fraction may be found which will express, to any degree of exactness, the multiple, parts, or part, that A is of B.

We will immediately establish this truth, and then no error can arise from saying, $\frac{a}{b} = \frac{c}{d}$, when $a : b :: c : d$, whether the magnitudes represented by a and b, or by c and d, are incommensurable or not. This may be shown as follows:—

Let m be contained in A more than p times, and less than $p + 1$ times; therefore A is greater than $p\,m$, and less than $(p + 1)\,m$; and let m be contained in B, q times exactly, $\therefore$ B $= q\,m$.

$$\therefore \quad \frac{A}{B} \sqsubset \frac{p\,m}{q\,m}, \text{ and } \sqsupset \frac{(p+1)\,m}{q\,m},$$

$$\text{or } \frac{A}{B} \sqsubset \frac{p}{q} \text{ and } \sqsupset \frac{p+1}{q}, \text{ or } \frac{p}{q} + \frac{1}{q};$$

therefore the fraction $\frac{p}{q}$ approaches nearer and nearer to $\frac{A}{B}$ the greater q is taken, because the greater q is, the less $\frac{1}{q}$ will be; and as B $= q\,m$, the greater q is taken the less m must be; therefore, by diminishing m, the difference between $\frac{A}{B}$ and $\frac{p}{q}$ may be made less than any that can be assigned. The same may be said of the fraction $\frac{C}{D}$, when C and D are incommensurable.

Therefore all the expositions of propositions respecting proportionals, founded on the principles, that $\frac{A}{B} = \frac{C}{D}$, when A : B :: C : D, or conversely, that A : B :: C : D when $\frac{A}{B} = \frac{C}{D}$, are correct, whether A and B and C and D be incommensurable or not.

DEFINITION X.

When three magnitudes are proportionals, the first is said to have to the third the duplicate ratio of that which it has to the second.

For example, if A, B, C, be continued proportionals, that is, A : B :: B : C, A is said to have to C the duplicate ratio of A : B;

$$\text{or } \frac{A}{C} = \text{the square of } \frac{A}{B}.$$

This property will be more readily seen of the quantities $a r^2$, $a r$, a,

$$\text{for } ar^2, : ar : ar : a;$$

$$\text{and } \frac{a r^2}{a} = r^2 = \text{the square of } \frac{a r^2}{a r} = r,$$

$$\text{or of } a,\ ar,\ ar^2;$$

$$\text{for } \frac{a}{a r^2} = \frac{1}{r^2} = \text{the square of } \frac{a}{a r} = \frac{1}{r}.$$

DEFINITION XI.

When four magnitudes are continual proportionals, the first is said to have to the fourth the triplicate ratio of that which it has to the second; and so on, quadruplicate, &c., increasing the denomination still by unity, in any number of proportionals.

For example, let A, B, C, D, be four continued proportionals, that is, A : B :: B : C :: C : D; A is said to have to D, the triplicate ratio of A to B;

$$\text{or } \frac{A}{D} = \text{the cube of } \frac{A}{B},$$

This definition will be better understood, and applied to a greater number of magnitudes than four that are continued proportionals, as follows :—

Let ar^3, ar^2, ar, a, be four magnitudes in continued proportion, that is,

$$ar^3 : ar^2 :: ar^2 : ar :: ar : a,$$

$$\text{then } \frac{ar^3}{r} = r^3 = \text{the cube of } \frac{ar^3}{ar^2} = r.$$

Or, let ar^5, ar^4, ar^3, ar^2, ar, a, be six magnitudes in proportion,

that is $ar^5 : ar^4 :: ar^4 : ar^3 :: ar^3 : ar^2 :: ar^2 : ar :: ar : a$,

$$\text{then the ratio } \frac{ar^5}{a} = r^5 = \text{the fifth power of } \frac{ar^5}{ar^4} = r.$$

Or, let a, ar, ar^2, ar^3, ar^4, be five magnitudes in continued proportion ;

$$\text{then } \frac{a}{ar^4} = \frac{1}{r^4} = \text{the fourth power of } \frac{a}{ar} = \frac{1}{r}.$$

DEFINITION A.

To know a compound ratio :—

When there are any number of magnitudes of the same kind, the first is said to have to the last of them the ratio compounded of the ratio which the first has to the second, and of the ratio which the second has to the third, and of the ratio which the third has to the fourth ; and so on, unto the last magnitude.

A B C D
E F G H K L
M N

For example, if A, B, C, D, be four magnitudes of the same kind, the first A is said to have to the last D the ratio compounded of the ratio of A to B, and of the ratio of B to C, and of the ratio of C to D ; or, the ratio of A to D is said to be compounded of the ratios of A to B, B to C, and C to D.

And if A has to B the same ratio which E has to F, and B to C the same ratio that G has to H, and C to D the same that K has to L; then, by this definition, A is said to have to D the ratio compounded of ratios which are the same with the ratios of E to F, G to H, and K to L. And the same thing is to be understood when it is more briefly expressed by saying, A has to D the ratio compounded of the ratios of E to F, G to H, and K to L.

In like manner, the same things being supposed; if M has to N the same ratio which A has to D, then, for shortness sake, M is said to have to N the ratio compounded of the ratios of E to F, G to H, and K to L.

This definition may be better understood from an arithmetical or algebraical illustration; for, in fact, a ratio compounded of several other ratios, is nothing more than a ratio which has for its antecedent the continued product of all the antecedents of the ratios compounded, and for its consequent the continued product of all the consequents of the ratios compounded.

Thus, the ratio compounded of the ratios of $2 : 3$, $4 : 7$, $6 : 11$, $2 : 5$,

is the ratio of $2 \times 4 \times 6 \times 2 : 3 \times 7 \times 11 \times 5$,

or the ratio of $96 : 1155$, or $23 : 385$.

And of the magnitudes A, B, C, D, E, F, of the same kind, $A : F$ is the ratio compounded of the ratios of $A : B$, $B : C$, $C : D$, $D : E$, $E : F$;

$$\text{for } A \times B \times C \times D \times E : B \times C \times D \times E \times F,$$

$$\text{or } \frac{A \times B \times C \times D \times E}{B \times C \times D \times E \times F} = \frac{A}{F}, \text{ or the ratio of } A : F.$$

PROP. F. THEO.

Ratios which are compounded of the same ratios are the same to one another.

Let $A : B :: F : G$,
$B : C :: G : H$,
$C : D :: H : K$,
and $D : E :: K : L$.

A	B.	C	D	E
F	G	H	K	L

Then the ratio which is compounded of the ratios of $A : B$, $B : C$, $C : D$, $D : E$, or the ratio of $A : E$, is the same as the ratio compounded of the ratios of $F : G$, $G : H$, $H : K$, $K : L$, or the ratio of $F : L$.

$$\text{For } \frac{A}{B} = \frac{F}{G},$$

$$\frac{B}{C} = \frac{G}{H},$$

$$\frac{C}{D} = \frac{H}{K},$$

$$\text{and } \frac{D}{E} = \frac{K}{L};$$

$$\therefore \quad \frac{A \times B \times C \times D}{B \times C \times D \times E} = \frac{F \times G \times H \times K}{G \times H \times K \times L},$$

$$\text{and } \therefore \quad \frac{A}{E} = \frac{F}{L},$$

or the ratio of $A : E$ is the same as the ratio of $F : L$.

The same may be demonstrated of any number of ratios so circumstanced.

Next, let $A : B :: K : L$,
$B : C :: H : K$,
$C : D :: G : H$,
$D : E :: F : G$.

Then the ratio which is compounded of the ratios of A : B, B : C, C : D, D : E, or the ratio of A : E, is the same as the ratio compounded of the ratios of K : L, H : K, G : H, F : G, or the ratio of F : L.

$$\text{For } \frac{A}{B} = \frac{K}{L},$$

$$\frac{B}{C} = \frac{H}{K},$$

$$\frac{C}{D} = \frac{G}{H},$$

$$\text{and } \frac{D}{E} = \frac{F}{G};$$

$$\therefore \frac{A \times B \times C \times D}{B \times C \times D \times E} = \frac{K \times H \times G \times F}{L \times K \times H \times G},$$

$$\text{and } \therefore \frac{A}{E} = \frac{F}{L},$$

or the ratio of A : E is the same as the ratio of F : L.

∴ Ratios which are compounded, &c.

PROP. G. THEO.

If several ratios be the same to several ratios, each to each, the ratio which is compounded of ratios which are the same to the first ratios, each to each, shall be the same to the ratio compounded of ratios which are the same to the other ratios, each to each.

A	B	C	D	E	F	G	H	P	Q	R	S	T
a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	V	W	X	Y	Z

$$\begin{array}{r|r|l}
\text{If } A : B :: a : b & \text{and } A : B :: P : Q & a : b :: V : W \\
C : D :: c : d & C : D :: Q : R & c : d :: W : X \\
E : F :: e : f & E : F :: R : S & e : f :: X : Y \\
\& \; G : H :: g : h & G : H :: S : T & g : h :: Y : Z
\end{array}$$

then $P : T = V : Z.$

$$\text{For } \frac{P}{Q} = \frac{A}{B} = \frac{a}{b} = \frac{V}{W},$$

$$\frac{Q}{R} = \frac{C}{D} = \frac{c}{d} = \frac{W}{X},$$

$$\frac{R}{S} = \frac{E}{F} = \frac{e}{f} = \frac{X}{Y},$$

$$\frac{S}{T} = \frac{G}{H} = \frac{g}{h} = \frac{Y}{Z};$$

$$\text{and } \therefore \; \frac{P \times Q \times R \times S}{Q \times R \times S \times T} = \frac{V \times W \times X \times Y}{W \times X \times Y \times Z},$$

$$\text{and } \therefore \; \frac{P}{T} = \frac{V}{Z},$$

$$\text{or } P : T = V : Z$$

∴ If several ratios, &c.

PROP. H. THEO.

If a ratio which is compounded of several ratios be the same to a ratio which is compounded of several other ratios, and if one of the first ratios, or the ratio which is compounded of several of them, be the same to one of the last ratios, or to the ratio which is compounded of several of them, then the remaining ratio of the first, or, if there be more than one, the ratio compounded of the remaining ratios, shall be the same to the remaining of the last; or, if there be more than one, to the ratio compounded of these remaining ratios.

A B C D E F G H
P Q R S T X

Let A : B, B : C, C : D, D : E, E : F, F : G, G : H, be the first ratios, and P : Q, Q : R, R : S, S : T, T : X, the other ratios; also, let A : H, which is compounded of the first ratios, be the same as the ratio of P : X, which is the ratio compounded of the other ratios; and, let the ratio of A : E, which is compounded of the ratios of A : B, B : C, C : D, D : E, be the same as the ratio of P : R, which is compounded of the ratios P : Q, Q : R.

Then the ratio which is compounded of the remaining first ratios, that is, the ratio compounded of the ratios E : F, F : G, G : H, that is, the ratio of E : H, shall be the same as the ratio of R : X, which is compounded of the ratios of R : S, S : T, T : X, the remaining other ratios.

$$\text{Because } \frac{A \times B \times C \times D \times E \times F \times G}{B \times C \times D \times E \times F \times G \times H} = \frac{P \times Q \times R \times S \times T}{Q \times R \times S \times T \times X},$$

$$\text{or } \frac{A \times B \times C \times D}{B \times C \times D \times E} \times \frac{E \times F \times G}{F \times G \times H} = \frac{P \times Q}{Q \times R} \times \frac{R \times S \times T}{S \times T \times H},$$

$$\text{and } \frac{A \times B \times C \times D}{B \times C \times D \times E} = \frac{P \times Q}{Q \times R},$$

$$\therefore \frac{E \times F \times G}{F \times G \times H} = \frac{R \times S \times T}{S \times T \times X},$$

$$\therefore \frac{E}{H} = \frac{R}{X},$$

$$\therefore E : H = R : X.$$

∴ If there be, &c.

PROP. K. THEO.

If there be any number of ratios, and any number of other ratios, such that the ratio which is compounded of ratios, which are the same to the first ratios, each to each, is the same to the ratio which is compounded of ratios, which are the same, each to each, to the last ratios—and if one of the first ratios, or the ratio which is compounded of ratios, which are the same to several of the first ratios, each to each, be the same to one of the last ratios, or to the ratio which is compounded of ratios, which are the same, each to each, to several of the last ratios—then the remaining ratio of the first; or, if there be more than one, the ratio which is compounded of ratios, which are the same, each to each, to the remaining ratios of the first, shall be the same to the remaining ratio of the last; or, if there be more than one, to the ratio which is compounded of ratios, which are the same, each to each, to these remaining ratios.

h k m n s
A B, C D, E F, G H, K L, M N, *a b c d e f g*
O P, Q R, S T, V W, X Y, *h k l m n p*
a b c d e f g

Let A : B, C : D, E : F, G : H, K : L, M : N, be the first ratios,

and O : P, Q : R, S : T, V : W, X : Y, the other ratios;

$$\begin{array}{rl} \text{and let } \mathrm{A} : \mathrm{B} &= a : b, \\ \mathrm{C} : \mathrm{D} &= b : c, \\ \mathrm{E} : \mathrm{F} &= c : d, \\ \mathrm{G} : \mathrm{H} &= d : e, \\ \mathrm{K} : \mathrm{L} &= e : f, \\ \mathrm{M} : \mathrm{N} &= f : g. \end{array}$$

Then, by the definition of a compound ratio, the ratio of $a : g$ is compounded of the ratios of $a : b$, $b : c$, $c : d$, $d : e$, $e : f$, $f : g$, which are the same as the ratio of $\mathrm{A} : \mathrm{B}$, $\mathrm{C} : \mathrm{D}$, $\mathrm{E} : \mathrm{F}$, $\mathrm{G} : \mathrm{H}$, $\mathrm{K} : \mathrm{L}$, $\mathrm{M} : \mathrm{N}$, each to each.

$$\begin{array}{rl} \text{Also, } \mathrm{O} : \mathrm{P} &= h : k, \\ \mathrm{Q} : \mathrm{R} &= k : l, \\ \mathrm{S} : \mathrm{T} &= l : m, \\ \mathrm{V} : \mathrm{W} &= m : n, \\ \mathrm{X} : \mathrm{Y} &= n : p. \end{array}$$

Then will the ratio of $h : p$ be the ratio compounded of the ratios of $h : k$, $k : l$, $l : m$, $m : n$, $n : p$, which are the same as the ratios of $\mathrm{O} : \mathrm{P}$, $\mathrm{Q} : \mathrm{R}$, $\mathrm{S} : \mathrm{T}$, $\mathrm{V} : \mathrm{W}$, $\mathrm{X} : \mathrm{Y}$, each to each.

$$\therefore \text{ by the hypothesis } a : g = h : p.$$

Also, let the ratio which is compounded of the ratios of $\mathrm{A} : \mathrm{B}$, $\mathrm{C} : \mathrm{D}$, two of the first ratios (or the ratios of $a : c$, for $\mathrm{A} : \mathrm{B} = a : b$, and $\mathrm{C} : \mathrm{D} = b : c$), be the same as the ratio of $\mathrm{a} : \mathrm{d}$, which is compounded of the ratios of $\mathrm{a} : \mathrm{b}$, $\mathrm{b} : \mathrm{c}$, $\mathrm{c} : \mathrm{d}$, which are the same as the ratios of $\mathrm{O} : \mathrm{P}$, $\mathrm{Q} : \mathrm{R}$, $\mathrm{S} : \mathrm{T}$, three of the other ratios.

And let the ratios of $\mathrm{h} : \mathrm{s}$, which is compounded of the ratios of $\mathrm{h} : \mathrm{k}$, $\mathrm{k} : \mathrm{m}$, $\mathrm{m} : \mathrm{n}$, $\mathrm{n} : \mathrm{s}$, which are the same as the remaining first ratios, namely, $\mathrm{E} : \mathrm{F}$, $\mathrm{G} : \mathrm{H}$, $\mathrm{K} : \mathrm{L}$, $\mathrm{M} : \mathrm{N}$; also, let the ratio of $\mathrm{e} : \mathrm{g}$, be that which is compounded of the ratios $\mathrm{e} : \mathrm{f}$, $\mathrm{f} : \mathrm{g}$, which are the same, each to each, to the remaining other ratios, namely, $\mathrm{V} : \mathrm{W}$, $\mathrm{X} : \mathrm{Y}$. Then the ratio of $\mathrm{h} : \mathrm{s}$ shall be the same as the ratio of $\mathrm{e} : \mathrm{g}$; or $\mathrm{h} : \mathrm{s} = \mathrm{e} : \mathrm{g}$.

For $\frac{A \times C \times E \times G \times K \times M}{B \times D \times F \times H \times L \times N} = \frac{a \times b \times c \times d \times e \times f}{b \times c \times d \times e \times f \times g}$,

and $\frac{O \times Q \times S \times V \times X}{P \times R \times T \times W \times Y} = \frac{h \times k \times l \times m \times n}{k \times l \times m \times n \times p}$,

by the composition of the ratios;

$$\therefore \frac{a \times b \times c \times d \times e \times f}{b \times c \times d \times e \times f \times g} = \frac{h \times k \times l \times m \times n}{k \times l \times m \times n \times p} \text{ (hyp.)},$$

or $\frac{a \times b}{b \times c} \times \frac{c \times d \times e \times f}{d \times e \times f \times g} = \frac{h \times k \times l}{k \times l \times m} \times \frac{m \times n}{n \times p}$,

but $\frac{a \times b}{b \times c} = \frac{A \times C}{B \times D} = \frac{O \times Q \times S}{P \times R \times T} = \frac{\mathrm{a \times b \times c}}{\mathrm{b \times c \times d}} = \frac{h \times k \times l}{k \times l \times m}$;

$$\therefore \frac{c \times d \times e \times f}{d \times e \times f \times g} = \frac{m \times n}{n \times p}.$$

And $\frac{c \times d \times e \times f}{d \times e \times f \times g} = \frac{\mathrm{h \times k \times m \times n}}{\mathrm{k \times m \times n \times s}}$ (hyp.),

and $\frac{m \times n}{n \times p} = \frac{\mathrm{e \times f}}{\mathrm{f \times g}}$ (hyp.),

$$\therefore \frac{\mathrm{h \times k \times m \times n}}{\mathrm{k \times m \times n \times s}} = \frac{\mathrm{ef}}{\mathrm{fg}},$$

$$\therefore \frac{\mathrm{h}}{\mathrm{s}} = \frac{\mathrm{e}}{\mathrm{g}},$$

$$\therefore \mathrm{h} : \mathrm{s} = \mathrm{e} : \mathrm{g}.$$

∴ If there be any number, &c.

PROP. L. THEO.

If $A : B :: C : D$, then $A^m : B^m :: C^m : D^m$,

and $\sqrt[n]{A} : \sqrt[n]{B} :: \sqrt[n]{C} : \sqrt[n]{D}$, and $A^{\frac{m}{n}} : B^{\frac{m}{n}} :: C^{\frac{m}{n}} : D^{\frac{m}{n}}$.

Because $A : B :: C : D$,

$$\frac{A}{B} = \frac{C}{D};$$

$$\therefore \quad \frac{A^2}{B^2} = \frac{C^2}{D^2}, \text{ or } A^2 : B^2 :: C^2 : D^2,$$

$$\text{and} \quad \frac{A^3}{B^3} = \frac{C^3}{D^3}, \text{ or } A^3 : B^3 :: C^3 : D^3,$$

$$\text{and } \therefore A^m : B^m :: C^m : D^m.$$

$$\text{Again, because } \frac{A}{B} = \frac{C}{D},$$

$$\therefore \quad \frac{\sqrt[n]{A}}{\sqrt[n]{B}} = \frac{\sqrt[n]{C}}{\sqrt[n]{D}}, \text{ or } \sqrt[n]{A} : \sqrt[n]{B} :: \sqrt[n]{C} : \sqrt[n]{D}.$$

$$\text{and } \therefore \quad \frac{A^{\frac{m}{n}}}{B^{\frac{m}{n}}} = \frac{C^{\frac{m}{n}}}{D^{\frac{m}{n}}}, \text{ or } A^{\frac{m}{n}} : B^{\frac{m}{n}} :: C^{\frac{m}{n}} : D^{\frac{m}{n}}.$$

SYNOPSIS OF THE DOCTRINE OF PROPORTION,

Algebraically and Arithmetically expressed.

	Let $A : B :: C : D$.		Let $2 : 3 :: 6 : 9$.
1.	$\frac{A}{B} = \frac{C}{D}$.	1.	$\frac{2}{3} = \frac{6}{9}$.
2.	$A \times D = B \times C$.	2.	$2 \times 9 = 3 \times 6 = 18$.
3.	$B : A :: D : C$.	3.	$3 : 2 :: 9 : 6$.
4.	$A : C :: B : D$.	4.	$2 : 6 :: 3 : 9$.
5.	$A + B : B :: C + D : D$.	5.	$2 + 3 : 3 :: 6 + 9 : 9$.
6.	$A + B : A :: C + D : C$.	6.	$2 + 3 : 2 :: 6 + 9 : 6$.
7.	$A \sim B : A :: C \sim D : C$.	7.	$2 \sim 3 : 2 :: 6 \sim 9 : 6$, or $1 : 2 :: 3 : 6$.
8.	$A \sim B : B :: C \sim D : D$.	8.	$2 \sim 3 : 3 :: 6 \sim 9 : 9$, or $1 : 3 :: 3 : 9$.
9.	$A+B : A \sim B :: C+D : C \sim D$.	9.	$3 + 2 : 2 \sim 3 :: 6 + 9 : 6 \sim 9$, or $5 : 1 :: 15 : 3$.

10. $nA : mB :: nC : mD.$	10. $4 \times 2 : 5 \times 3 :: 4 \times 6 : 5 \times 9,$ or $8 : 15 :: 24 : 45;$ $n = 4, m = 5.$
11. $nA : B :: nC : D.$	11. $3 \times 2 : 3 :: 3 \times 6 : 9,$ or $6 : 3 :: 18 : 9;$ here $n = 3.$
12. $A : nB :: C : nD.$	12. $2 : 5 \times 3 :: 6 : 5 \times 9,$ or $2 : 15 :: 6 : 45;$ $n = 5.$
13. $A^2 : B^2 :: C^2 : D^2.$	13. $2^2 : 3^2 :: 6^2 : 9^2,$ or $4 : 9 :: 36 : 81.$
14. $A^3 : B^3 :: C^3 : D^3.$	14. $2^3 : 3^3 :: 6^3 : 9^3,$ or $8 : 27 :: 216 : 729.$
15. $A^n : B^n :: C^n : D^n.$	15. $2^5 : 3^5 :: 6^5 : 9^5,$ or $32 : 243 :: 7776 : 59049;$ here $n = 5.$
16. If $A : B :: C : D$, and if D be the greatest of the four, A must be the least, and $A + D \sqsubset B + C.$	16. If $2 : 3 :: 6 : 9,$ then $2 + 9 \sqsubset 6 + 3.$
17. $\frac{nA}{m} : \frac{pB}{q} :: \frac{nC}{m} : \frac{pD}{q}.$	17. $\frac{4 \times 2}{5} : \frac{7 \times 3}{8} :: \frac{4 \times 6}{5} : \frac{7 \times 9}{8},$ or $\frac{8}{5} : \frac{21}{8} :: \frac{24}{5} : \frac{63}{8};$ $n = 4, m = 5, p = 7,$ and $q = 8.$
18. If $a : b = c : d = e : f = g : h,$ $a : b = a + c + e + g : b + d + f + h.$	18. $2 : 3 = 6 : 9 = 12 : 18 = 8 : 12,$ $2 : 3 = 2 + 6 + 12 + 8 : 3 + 9 + 18 + 12;$ $2 : 3 = 28 : 42.$

19. If $a : b :: c : d$,
$e : f :: g : h$,
$k : l :: m : n$,
$o : p :: q : r$;
then $aeko : bflp :: cgmq : dhnr$,

$$\text{or } \frac{aeko}{bflp} = \frac{cgmq}{dhnr}.$$

19. $2 : 1 :: 30 : 15$,
$8 : 3 :: 16 : 6$,
$3 : 2 :: 6 : 4$,
$5 : 3 :: 10 : 6$;
then $2 \times 8 \times 3 \times 5 : 1 \times 3 \times 2 \times 3 ::$
$30 \times 16 \times 6 \times 10 : 15 \times 6 \times 4 \times 6$,
or $240 : 18 :: 28800 : 2160$.

20. If $\frac{A}{B} \sqsubset \frac{C}{D}$,
$A : B \sqsubset C : D$;
or $C : D \sqsupset A : B$.

20. $\frac{3}{4} \sqsubset \frac{2}{3}$,
$3 : 4 \sqsubset 2 : 3$,
or $2 : 3 \sqsupset 3 : 4$.

21. a, b, c, d, e, f,
a', b', c', d', e', f'.
If $a : b = a' : b', b : c = b' : c'$,
$c : d = c' : d', d : e = d' : e', e : f = e' : f'$;
then $a : f = a' : f'$.

21. $4, 2, 3, 12$,
$8, 4, 6, 24$,
$4 : 2 = 8 : 4,\ 2 : 3 = 4 : 6$,
$3 : 12 = 6 : 24$,
$4 : 12 = 8 : 24$.

21. a, b, c, d,
a', b', c', d'.
If $a : b = c' : d', b : c = b' : c'$,
$c : d = a' : b'$,
then $a : d = a' : d'$.

21. $3, 6, 2, 7$,
$\frac{6}{7}, 3, 1, 2$,
$3 : 6 = 1 : 2,\ 6 : 2 = 3 : 1$,
$2 : 7 = \frac{6}{7} : 3$, then $3 : 7 = \frac{6}{7} : 2$.

22. If $A^n : B^n :: C^n : D_n$,
we can immediately infer that $A + B : A \sim B :: C + D : C \sim D$;
or, if $\frac{nA}{m} : \frac{pB}{q} :: \frac{nC}{m} : \frac{pD}{q}$,
we may infer $A^2 : B^2 :: C^2 : D^2$;
or any other property respecting four proportionals.

SUPPLEMENT I.

ON COMMENSURABLE AND INCOMMENSURABLE MAGNITUDES.

MAGNITUDES which are multiples of any other magnitude are said to be commensurable, and this other magnitude is called a common measure.

Magnitudes which have no common measure, that is, magnitudes which are not multiples of any other magnitude, are said to be incommensurable.

Numbers have always a common measure, for they can always be measured by a unit. When they can only be measured by a unit, they are called prime numbers: numbers are said to be prime to each other when no number except a unit will measure them. Thus, 43, 47, 53, 61, are prime numbers; and 5, 8, 51, 77, are prime to each other, for no number except a unit is contained an exact number of times in all: yet 8 can be measured by 1, 2, and 4; 51 by 1, 17, and 3; and 77 by 1, 7, and 11.

PROP. I. THEO.

If a magnitude measure each of two others, it will also measure their sum and difference.

Let A measure B and C separately, it will also measure $B + C$ and $B - C$, allowing B to be the greater of B and C; unless one of the magnitudes B or C be greater than the other, the difference cannot be measured by A, as it is 0.

Let A be contained in B, m times exactly, $\therefore B = mA$;
and A to be contained in C, n times exactly, $\therefore C = nA$;
it is evident that n must be less than m, for B is greater than C.

$$\therefore B + C = mA + nA = (m + n)A;$$

$$\text{and } B - C = mA - nA = (m - n)A.$$

Therefore A is contained in $B + C$, $m + n$ times, and in $B - C$, $m - n$ times; consequently A measures the sum and difference of B and C.

It will immediately follow, that if A measures B + C and C, it will also measure B, for B is the difference of B + C and C; or if A measures B + C and B, it will measure C for the same reason. And if A measures B − C and C, it will also measure B, for B is the sum of C and B − C; it is evident also, if A measures B − C and B, it will measure C, for C is the difference between B and B − C.

PROP. II. PROB.

Two magnitudes of the same kind being given to find their greatest common measure, if it be possible.

Let A and B be the two magnitudes, and A less than B; it is required to find the greatest magnitude that will measure them both.

Let p be the greatest number of times that A is contained in B; take p A from B, and let the remainder be C, which must be less than A. Let q be the greatest number of times that C is contained in A; from A take q C, and let the remainder be D. This operation is to be continued, always dividing the preceding divisor by the last remainder, till nothing remains; the remainder, which divides the preceding one exactly, is the greatest common measure.

$$\begin{array}{l}
\mathrm{A}\,)\,\mathrm{B}\,(\,p \\
\quad \underline{p\,\mathrm{A}} \\
\quad \mathrm{C}\,)\,\mathrm{A}\,(\,q \\
\qquad \underline{q\,\mathrm{C}} \\
\qquad \mathrm{D}\,)\,\mathrm{C}\,(\,r \\
\qquad\quad \underline{r\,\mathrm{D}} \\
\qquad\quad \mathrm{E}\,)\,\mathrm{D}\,(\,s \\
\qquad\qquad \underline{s\,\mathrm{E}} \\
\qquad\qquad \&\mathrm{c.}
\end{array}$$

It must be observed, that A, B, C, D, E, &c., represent magnitudes of the same kind, and p, q, r, s, &c., whole numbers.

First, we shall prove that any magnitude which measures A and B, will also measure the remainders C, D, E, &c. Let M measure both A and B, it will measure any multiple of A ; ∴ it will measure p A; and, as it measures B, it will also measure B − p A or C (prop. i.); and, as it measures C, it will also measure q C ; and M measures A, ∴ it will measure A − q C or D; as it measures D, it will measure r D; and it has been shown to measure C, ∴ it will measure C − r D or E. Therefore any magnitude, M, which measures A and B, will measure all the remainder, C, D, E, &c., unto the last.

Now we have shown that every common measure of A and B, will measure the last remainder; and it is evident the greatest magnitude that will measure the last remainder is itself; therefore we have only to prove that the last remainder measures A and B, to show that the last remainder is the greatest common measure of A and B. This may be shown as follows :—

Let E be the last remainder, or that E is contained in D an exact number of times; $\therefore$ E will measure D, and it will also measure itself and r D; $\therefore$ E will measure r D + E or C; and, as E will measure C, it will also measure q C; and it has been shown to measure D, $\therefore$ E will measure q C + D or A. Again, because E measures A, it will measure p A; and it has been shown that E measures C, $\therefore$ E measures p A + C or B; $\therefore$ the last remainder measures A and B. If there be no such thing as a last remainder, that is, when the above process does not terminate, the magnitudes have no common measure, or they are incommensurable.

PROP. III. THEO.

If two numbers be prime to each other, they are the least two numbers in that proportion.

Let $\frac{B}{A}$ be a fraction where A is prime to B, and if it be possible let this fraction be equal to another, $\frac{B'}{A'}$, reduced to its lowest terms, by dividing by the greatest common measure; and if it be possible let B be greater than B′, and A greater than A′, at the same time, $\frac{B}{A} = \frac{B'}{A'}$, or $\frac{A}{B} = \frac{A'}{B'}$.

Divide B by A, and B′ by A′, as in the last problem.

$$\begin{array}{lll} A\,)\,B\,(\,m & \qquad\qquad & A'\,)\,B'\,(\,m \\ \;\;mA & & \;\;mA' \\ \hline \;\;C\,)\,A\,(\,n & & \;\;C'\,)\,A'\,(\,n \\ \;\;\;\;nC & & \;\;\;\;nC' \\ \hline \;\;\;\;D\,)\,C\,(\,r & & \;\;\;\;D'\,)\,C'\,(\,r \\ \;\;\;\;\;\;rD & & \;\;\;\;\;\;rD' \\ \hline \;\;\;\;\;\;E & & \;\;\;\;\;\;E' \end{array}$$

Because $\frac{B}{A} = \frac{B'}{A'}$, the first quotients, m, m, are equal; and since $\frac{B}{A} = m + \frac{C}{A}$, and $\frac{B'}{A'} = m + \frac{C'}{A'}$, $\therefore$ $\frac{C}{A} = \frac{C'}{A'}$ or $\frac{A}{C} = \frac{A'}{C'}$; and because A is greater than A′, C must be greater than C′. Again, because $\frac{A}{C} = \frac{A'}{C'}$, the next quotients, n, n, are equal; and as $\frac{A}{C} = n + \frac{D}{C}$, and $\frac{A'}{C'} = n + \frac{D'}{C'}$, $\therefore$ $\frac{D}{C} = \frac{D'}{C'}$; and because C is greater than C′, D must be greater than D′. In the same manner it may be shown that E is greater than E′, &c.; ∴ a remainder in the first division must always be greater than the one similarly circumstanced in the latter; ∴ one of the remainders in the latter division will become unity sooner than in the former. Let $E' = 1$; then E must be greater than 1; and since $\frac{D}{E} = \frac{D'}{E'}$, ∴ $\frac{D}{E} = D'$; ∴ E, which is greater than 1, measures D, ∴ E must measure both A and B, according to the last proposition. But this is contrary to the hypothesis, for A and B were supposed prime to each other; consequently we may infer that when A is prime to B they are the least in the proportion, and when there are other numbers in the same proportion they must be multiples of these.

PROP. IV. THEO.

If two numbers be each prime to another number, their product will be prime to that number.

Let A and B be each prime to C;
then $A \times B$ will be prime to C:
for, if not, let $A \times B = m \times P$, and $C = m \times Q$.
Since A and B are both prime to C, they are also prime to $m \times Q$,
and ∴ to m.

Now, $A \times B = m \times P$; divide each by m B, and we have, $\frac{A}{m} = \frac{P}{B}$; but as it has been shown that A is prime to m, therefore, by the last proposition, P must be a multiple of A, and B a multiple of m. But it has also been shown that B is prime to m, which is absurd. Therefore, if two numbers be prime to another number, their product will be prime to that number.

PROP. V. THEO.

If two numbers be prime to each other, one of them is prime to the square, cube, &c., of the other; and any power of one of them will be prime to any power of the other.

Let A be prime to B;
then A^2 is prime to B,
A^3 to B,
&c., &c.

For A and A are prime to B, therefore, by the last proposition, A^2 is prime to B; and as A and A^2 are each prime to B, A^3 must be prime to B (prop. iv.), and so on, it may be shown that A^n is prime to B.

Again, because B and B are each prime to A^n, B^2 is prime to A^n; and as B and B^2 are each prime to A^n, ∴ B^3 is prime to A^n; and in the same manner it may be shown that B^m is prime to A^n.

It can also be readily shown that if

A be prime to B, D, F, &c.
C „ to D, F, B, &c.
E „ to F, B, D, &c.
&c., &c.

that A × C × E, &c., will be prime to B × D × F, &c.

PROP. VI. THEO.

If one magnitude contain another any number of times, and leave a remainder, such that the greater of the two magnitudes is to the smaller as the smaller is to the remainder, then the two magnitudes will be incommensurable.

Let A, the greater of two magnitudes, contain the smaller, B, any number of times, leaving the magnitude C less than B, such that A : B :: B : C; then A and B are incommensurable, or have not a common measure.

Let the successive remainders, in finding the common measure of A and B, be C, D, E, &c. (prop. ii.)

Because A : B :: B : C, and B does not measure A, ∴ C cannot measure B; and yet A contains B as often as B contains C. Let sB be the greatest multiple of B which is contained in A; and let tC be an equimultiple of C, which must therefore be the greatest multiple of C contained in B, or in other terms $s = t$;

$$\begin{array}{l} B\,)\,A\,(\,s \\ \quad\underline{sB} \\ \quad C\,)\,B\,(\,t \\ \qquad\underline{tC} \\ \qquad D\,)\,C\,(\,u \\ \qquad\quad\underline{uD} \\ \qquad\quad E, \text{\&c.} \end{array}$$

$$\begin{gathered} \therefore\ A : sB :: B : tC, \\ \therefore\ A : A - sA :: B : B - tC, \\ \therefore\ A : C :: B : D, \\ \therefore\ A : B :: C : D, \\ \text{but } B : C :: A : B, \\ \therefore\ B : C :: C : D. \end{gathered}$$

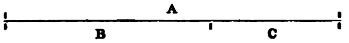
Now it is evident that D cannot measure C, because C cannot measure B. Pursuing the same course of reasoning as before, we may readily show that

$C : D :: D : E$, and that E cannot measure D, because D does not measure C. Consequently the process for finding the common measure of A and B is interminable, ∴ the magnitudes are incommensurable.*

PROP. VII. THEO.

If a line be divided in extreme and mean proportion, the two parts will be incommensurable.

A line, A, is said to be divided in extreme and mean proportion, when it is divided into two parts, B and C, such that $A : B :: B : C$.



Allowing this to be the case, A and B are incommensurable, by the last proposition; and therefore B and C are also incommensurable.

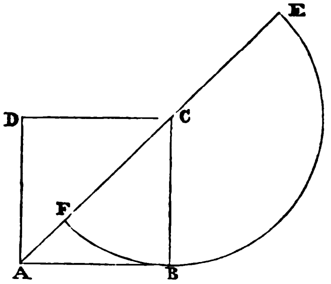
PROP. VIII. THEO.

The diagonal and side of a square are incommensurable.

Let A B C D be a square; the diagonal A C is incommensurable with its side A B.

Produce A C, and from the point C, as a centre, with the radius C B, describe the semicircle F B E.

The angle A B C being right, A B touches the circle, therefore (36 B. iii.) $AE \times AF = AB^2$; or $AE : AB :: AB : AF$.



And as A E contains A B twice, and leaves a remainder, A F; ∴ A E and A B are incommensurable; hence A C and A B are incommensurable, for if these had a common measure, the same would also measure their sum, A E, which is incommensurable with A B. Therefore the diagonal of a square is incommensurable with its side.

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* This proposition, with some alteration, is taken from Professor Young's "Elements of Geometry," Prop. xix, B. v.

Because there are no two numbers that cannot be measured by another (at least by a unit); and as the demonstration of proposition vii. shows that the segments of a line, divided in extreme and mean proportion, are incommensurable, or cannot have a common measure; the lengths of the segments cannot be perfectly expressed by numbers. Again, from the demonstration of the last proposition (viii), it is evident that the side and diagonal of a square cannot be fully expressed by numbers, for they are also incommensurable; and in fact no incommensurable magnitudes can be fully expressed by numbers, or if one of them can be so expressed the others cannot. Hence it might appear, that some very great error should arise if numbers were employed to represent such magnitudes; or that the application of numbers is limited; or that they are not capable of that refined accuracy of expression to which our geometrical notions of things have arrived. The truth is, not one of these conjectures is correct; for we might as well question the accuracy or capability of decimal arithmetic, because the exact decimals of $\frac{1}{3}$, $\frac{2}{7}$, $\frac{5}{9}$, &c., cannot be wholly expressed. But it is well known that the decimal values of these or any other fractions can be obtained to any degree of accuracy that may be assigned; the same may be said of the application of numbers to incommensurable magnitudes.

To give an idea of the sufficiency of numbers to express incommensurable magnitudes, let us suppose the length of a right line to be 100 millions of miles, and that it is required to express, as near as possible, in numbers, the lengths of the segments of a line of such length, divided in extreme and mean proportion; or to determine, as near as possible, the point which separates the segments. In a very short time we can find that the point which separates the segments is situated in a dot, much less than a period in this print, the distance of which from either of the ends of the line is exactly known—aye, numbers are able to determine that the point which separates the segments is situated in a dot, so small that the greatest microscopic power known could not render it visible; and yet the distance of this dot from either end of the line is exactly known. Next, let us take a square, the side of which let us suppose to be the length of the line taken above, the length of the diagonal of this square can be readily expressed in numbers, so that it will not differ from the true diagonal the breadth of a single hair. The difference between the true diagonal and that which may be expressed by numbers is so small that it cannot be named. And numbers are capable of expressing all other magnitudes, whether they be incommensurable or not, with the same accuracy. Hence numbers cannot be said to be defective either in point of accuracy or power of expression.

SUPPLEMENT II.

ON VARYING AND DEPENDING QUANTITIES.

As the theory of varying quantities is so useful, and so dependant on the doctrine of proportion, at the same time calculated to extend the student's views of that doctrine, a supplement, containing demonstrations of some of the most useful propositions of the theory of variable quantities, must in a great measure contribute to the completion of a work like the present.

In the theory of variable quantities, although only two terms in each proportion are expressed, yet the learner would do well, in such cases, to imagine the other two, for most of our conclusions are come to by considering the properties of proportionals.

DEFINITIONS.

I. A quantity, A, is said to vary directly as another, B; when one of them is changed, the other is changed in the same proportion.

Let A be changed to any other value, a; at the same time B becomes b.

Then, if $A : a :: B : b$,

A is said to vary directly as B.

The expression $A \propto B$ signifies, that A varies as B, and is read as such.

Ex.—If the altitude of a triangle do not vary, the area varies as the base. Suppose the altitude of a triangle to be $2\,a$, and the base B, the area will be $a\,B$; let B change to any other value, b, then the area will be $a\,b$. Then it is evident that $a\,B : a\,b :: B : b$; $\therefore$ the areas vary as their bases.

II. A is said to vary inversely as B, when A cannot be changed in any manner but $\frac{1}{B}$, or the reciprocal of B, is changed in the same proportion.

A varies inversely as B, is thus expressed, $A \propto \frac{1}{B}$.

This signifies that if A be changed to a, and B to b,

then $A : a :: \frac{1}{B} : \frac{1}{b}$; or $A : a :: b : B$.

Ex.—If the area of a triangle be given, the base varies inversely as the perpendicular altitude. Let A and a be the altitudes, and 2 B and 2 b the bases; ∴ the areas will be $A \times B$ and $a \times b$, as the area is supposed to be the same in both cases; ∴ $A \times B = a \times b$, and consequently $A : a :: b : B$, or $A : a :: \frac{1}{B} : \frac{1}{b}$, or $A : a :: \frac{1}{2B} : \frac{1}{2b}$,

III. One quantity, A, is said to vary as two others, B and C, jointly (thus written, $A \propto B \times C$); if the former, A, be changed in any manner, the product of the other two, B and C, is changed in the same proportion.

That is, let A be changed to any other value, a, and $B \times C$ changed to $b \times c$, so that $A : a :: B \times C : b \times c$; then A is said to vary jointly, as B and C.

Ex.—The area of a triangle varies as its base and perpendicular jointly: for let A, B, and P, represent the area, base, and perpendicular of a triangle, respectively; and a, b, and p, those of another; then $B \times P = 2A$, and $b \times p = 2a$;

$$\therefore \frac{B \times P}{b \times p} = \frac{2A}{2a} = \frac{A}{a};$$

$$\text{and } \therefore A : a :: B \times P : b \times p.$$

IV. A is said to vary directly as B, and inversely as C (written thus, $A \propto \frac{B}{C}$), when A cannot be changed in any manner, but $B \times \frac{1}{C}$, or B multiplied by the reciprocal of C, is changed in the same proportion.

That is, let A, B, and C be changed to a, b, and c; and let $A : a :: \frac{B}{C} : \frac{b}{c}$, then A is said to vary directly as B, and inversely as C.

Ex.—The base of a triangle varies as the area directly, and as the perpendicular altitude inversely.

Let A, B, and P, be the area, base, and perpendicular of a triangle, and let them be changed to a, b, and p, respectively; then we have, as before, $B \times P = 2A$, and $b \times p = 2a$;

$$\therefore \frac{B \times P}{b \times p} = \frac{2A}{2a} = \frac{A}{a};$$

by multiplying both sides by $\frac{p}{P}$, we have

$$\frac{B \times P \times p}{b \times p \times P} = \frac{A \times p}{a \times A},$$

$$\text{or } \frac{B}{b} = \frac{A}{P} \times \frac{p}{a};$$

$$\therefore B + b = \frac{A}{B} + \frac{a}{p},$$

$$\text{and } \therefore B : b :: \frac{A}{P} : \frac{a}{p}.$$

PROP. I. THEO.

If A varies as B, and B as C, then A varies as C.

In this proposition and the following, a, b, c, &c., represent corresponding values of A, B, C, &c.; supposing A, B, C, &c., to vary.

Let $A : a :: B : b$, and $B : b :: C : c$;

$\therefore A : a :: C : c$ (11 B. v.);

$\therefore$ if $A \propto B$, and $B \propto C$, $\therefore A \propto C$.

In the same manner it may be shown, if $A \propto B$, and $B \propto \frac{1}{C}$, then $A \propto \frac{1}{C}$.

PROP. II. THEO.

If $A \propto C$, and $B \propto C$, then $A \pm B \propto C$; and $\sqrt{AB} \propto C$.

By the supposition $A : a :: C : c :: B : b$,

$\therefore A : a :: B : b$ (11 B. v.);

$\therefore A : B :: a : b$ (alt.);

$\therefore A \pm B : B :: a \pm b : b$ (compo.);

and $\therefore A \pm B : a \pm b :: B : b$ (alt.).

Again, because $A : a :: C : c$,

and $B : b :: C : c$,

$\therefore AB : ab :: C^2 : c^2$ (L. B. v.),

and $\therefore \sqrt{AB} : \sqrt{ab} :: C : c$ (L. B. v.);

consequently $\sqrt{AB} \propto C$.

PROP. III. THEO.

If $A \propto B$, then $A \propto mB$, and $A \propto \frac{B}{m}$; m being any number whatever.

$$\text{Because } A : a :: B : b,$$
$$\therefore A : a :: mB : mb;$$
$$\therefore A \propto mB.$$
$$\text{Again, because } A : a :: B : b,$$
$$\therefore A : a :: \frac{B}{m} : \frac{b}{m};$$
$$\therefore A \propto \frac{B}{m}.$$

PROP. IV. THEO.

If A vary as B. A is always equal some constant multiple or submultiple of B.

$$\text{Because } A : a :: B : b,$$
$$\therefore A : a :: mB : mb;$$
$$\text{and } \therefore A : mB :: a : mb.$$

Now, if m be taken so that $A = mB$, then must $a = mb$, in all cases; and if any constant values of A and B be known, m may be found:

$$\text{For } A = mB, \therefore m = \frac{A}{B}.$$

PROP. V. THEO.

If $A \propto B$, then $A^n \propto B^n$; n being any number—whole or fractional.

$$\text{For } A : a :: B : b \text{ (hyp.)}$$
$$\therefore A^n : a^n :: B^n : b^n \text{ (L. B. v.)}$$
$$\therefore A^n \propto B^n.$$

PROP. VI. THEO.

If one quantity vary as another, and each of them be multiplied or divided by any number, the products or quotients will vary as each other.

Let A vary as B, and let T be any number—constant or variable.

$$\text{Then, by the supposition, } A : a :: B : b,$$
$$\text{and } T : t :: T : t;$$
$$\therefore AT : at :: BT : bt.$$
$$\text{or } AT \propto BT.$$
$$\text{Again, } \frac{A}{T} : \frac{a}{t} :: \frac{B}{T} : \frac{b}{t},$$
$$\text{or } \frac{A}{T} \propto \frac{B}{T}.$$

Cor. If $A \propto B$, by dividing both by B, we have $\frac{A}{B} \propto 1$, that is, $\frac{A}{B}$ varying as one, but one is constant, therefore $\frac{A}{B}$ is constant.

PROP. VII. THEO.

If $A \propto B \times C$, then B varies directly as A, and inversely as C; and C varies directly as A, and inversely as B.

Because $A \propto B \times C$, by dividing by C, we have $\frac{A}{C} \propto B$, or B varying inversely as C, and directly as A. The other part of the proposition may be readily proved, by dividing by B.

PROP. VIII. THEO.

If the product of two quantities be invariable, these quantities vary inversely as each other.

$$\text{Let } B \times P \text{ be constant, or } B \times P \propto 1;$$
$$\therefore B \propto \frac{1}{p} \text{ and } p \propto \frac{1}{B}.$$

PROP. IX. THEO.

If four quantities be always proportionals, and one of them be invariable, we may find how the others vary.

$$\text{Let } a : b :: c : d, \text{ and } a \text{ invariable};$$
$$\text{then we can find how } b, c, \text{ and } d \text{ vary};$$
$$\text{For } ad = bc, \therefore ad \propto bc,$$
$$\text{and } \therefore d \propto bc \text{ (prop. iii)}.$$

PROP. X. THEO.

If $A \propto B$, and $C \propto D$, then $AC \propto BD$.

For $A : a :: B : b$,
and $C : c :: D : d$;
$\therefore\ AC : ac :: BD : bd$,
and $\therefore\ AC \propto BD$.

PROP. XI. THEO.

When the increase or decrease of one quantity depends on the increase or decrease of two others, and it appears that either of these latter be invariable, the first varies as the other; when they both vary, the first varies as their product.

Let $S \propto V$, when T is constant,
and $S \propto T$, V is constant.

When neither T nor V is constant, $S \propto TV$.

If S be changed to s, let T be changed to t,
so that $S : s :: T : t$, allowing V to be constant.

Again, let s be changed to s', and V to v,
so that $s : s' :: V : v$, T being constant;
$\therefore\ Ss : ss' :: TV : tv$;
or $S : s' :: TV : tv$;
$\therefore\ S \propto TV$.

This will be readily seen, if S be considered as a variable space, passed over by a body moving with a variable uniform velocity in a given time, T; and again, if S be considered as a variable space, passed over by a body moving with a given uniform velocity, V, in a variable time, T; then it is clear, when the time and velocity both vary, the space varies as the time multiplied by the velocity.

PROP. XII. THEO.

If $A \propto P$, when Q, R, S, &c., are constant; and if $A \propto Q$, when P, R, S, &c., are constant; and if $A \propto R$, when P, Q, R, S, &c., all become variable; $A \propto P \times Q \times R \times S$, &c.

The truth of this proposition may be readily shown by employing the process used in the demonstration of the last proposition.

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ERRATA.

In Definition XIV, page 16, *for* invertando, *read* invertendo; and *for* invertion, *read* inversion.

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